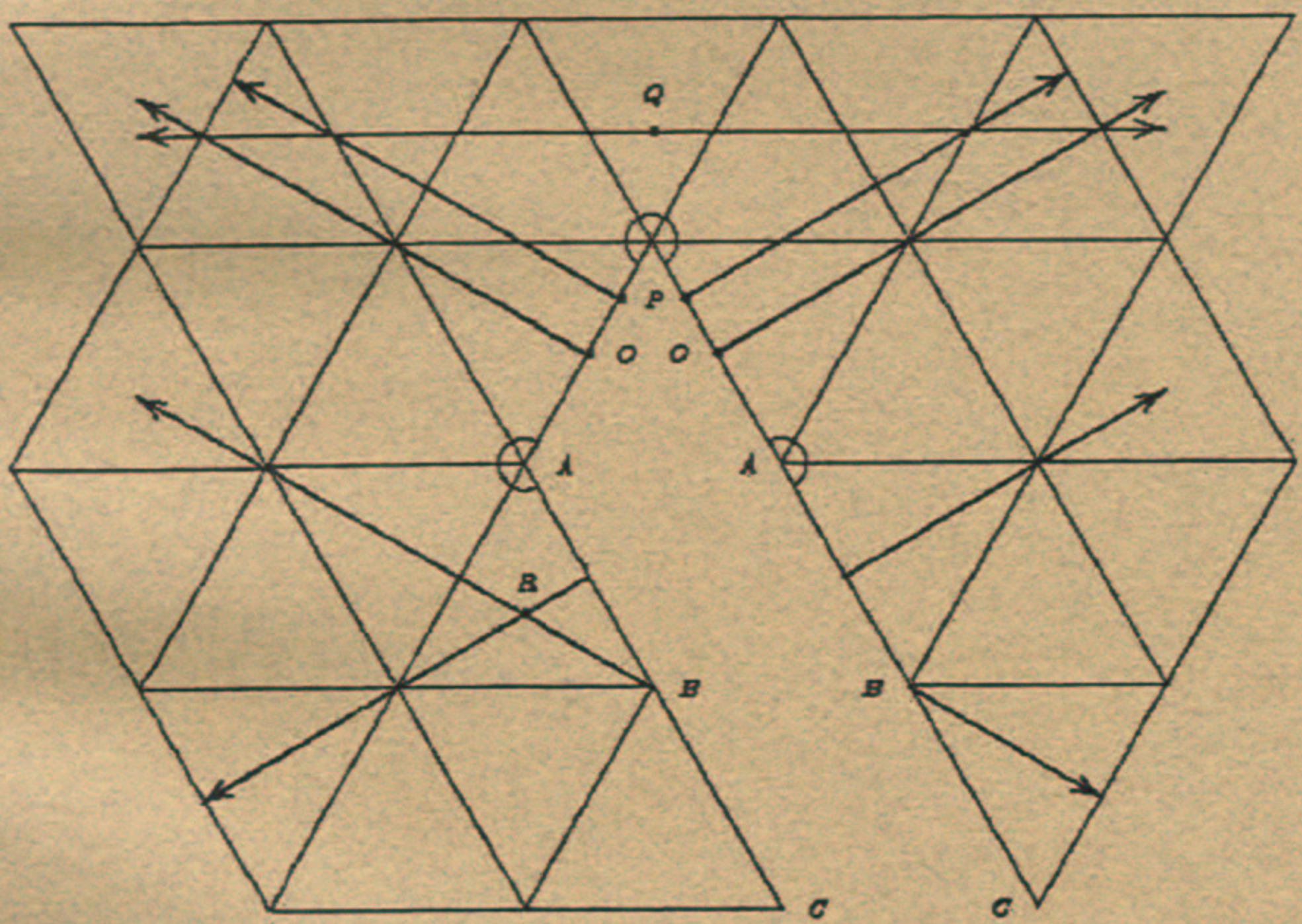
JACK ALLEN FENG LIU DRAGOŞ CONSTANTINESCU

SMARANDACHE NOTIONS
(book series, Vol. 13)



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The graph on the first cover represents a Howard Model for a Smarandache Geometry [see p. 8].

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FOREWARD

Papers concerning any of the Smarandache type functions, sequences, integer algorithms, paradoxes, Non-Euclidean geometries, conjectures, open problems, neutrosophic logic/set/probability, etc. have been selected for this volume.

The Editors

Partially Paradoxist Smarandache Geometries

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Abstract: A paradoxist Smarandache geometry combines Euclidean, hyperbolic, and elliptic geometry into one space along with other non-Euclidean behaviors of lines that would seem to require a discrete space. A class of continuous spaces is presented here together with specific examples that exhibit almost all of these phenomena and suggest the prospect of a continuous paradoxist geometry.

Introduction

Euclid's parallel postulate can be formulated to say that given a line l and a point P not on l, there is exactly one line through P that is parallel to l. An axiom is said to be **Smarandachely denied**, if it, or one of its negations, holds in some instances and fails to hold in others within the same space. For example, Euclid's parallel postulate would be Smarandachely denied in a geometry that was both Euclidean and non-Euclidean, or non-Euclidean in at least two different ways. A **Smarandache geometry** is one that has at least one Smarandachely denied axiom, and a paradoxist Smarandache geometry, to be described later, denies Euclid's parallel postulate in a somewhat exhaustive way.

Euclid's parallel postulate does not hold in the standard non-Euclidean geometries, the hyperbolic geometry of Gauss, Lobachevski, and Bolyai and the elliptic geometry of Riemann. These are special cases of the two-dimensional manifolds of Riemannian geometry. Here the three types of geometry are characterized by the Gauss curvature, negative curvature for hyperbolic, zero curvature for Euclidean, and positive curvature for elliptic. In general, the curvature may vary within a particular Riemannian manifold, so it is possible that the geodesics, the straightest possible curves, will behave like the lines of Euclidean geometry in one region and like the lines of hyperbolic or elliptic geometry in another. We would expect, therefore, to find geometries among the Riemannian manifolds that Smarandachely deny Euclid's parallel postulate. The models presented here will suggest specific examples, but explicit descriptions would be far from trivial.

We will bypass the computational complexities of Riemannian manifolds by turning to a class of geometric spaces that we will call Smarandache manifolds or S-manifolds. S-manifolds are piecewise linear manifolds topologically, and they have geodesics that exhibit elliptic, hyperbolic, and Euclidean behavior similar to those in Riemannian geometry, but that are much easier to construct and describe.

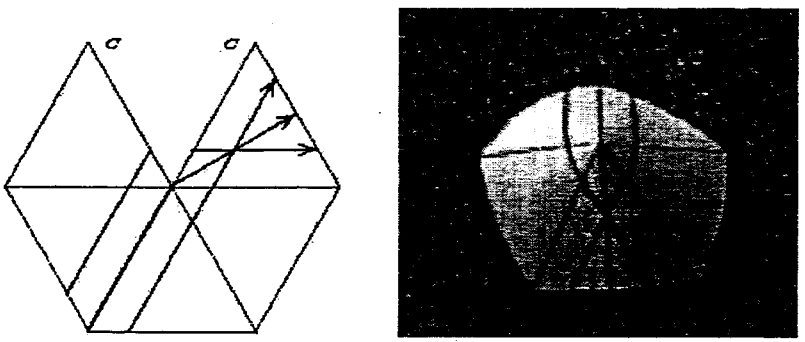
The idea of an S-manifold is based on the *hyperbolic paper* described in [2] and credited to W. Thurston. There, the negative curvature of the hyperbolic plane is visualized by taping together seven triangles made of paper (see Figures 2a and 2b). Squeezing seven equilateral triangles around a vertex, instead of the usual six seen in a tiling of the plane, forces the paper into a flat saddle shape with the negative curvature concentrated at the center vertex. By utilizing these "curvature singularities," our S-manifolds can be flat (i.e., Euclidean) everywhere else.

Smarandache manifolds

A **Smarandache manifold** (or **S-manifold**) is a collection of equilateral triangular disks (triangles) where every edge is shared by exactly two triangles, and every vertex is shared by five, six, or seven triangles. The points of the manifold are those of the triangular disks, including all the interior points, edge points, and vertices. Lines (geodesics) in the manifold are those piecewise linear curves with the following properties. They are straight in the Euclidean sense within each triangular disk and pair of adjacent triangular disks (since two triangles will lie flat in the plane). Across a vertex, a line will make two equal angles (two 150° angles for five triangles, two 180° angles for six triangles, and two 210° angles for seven).

Elliptic Vertices — five triangles

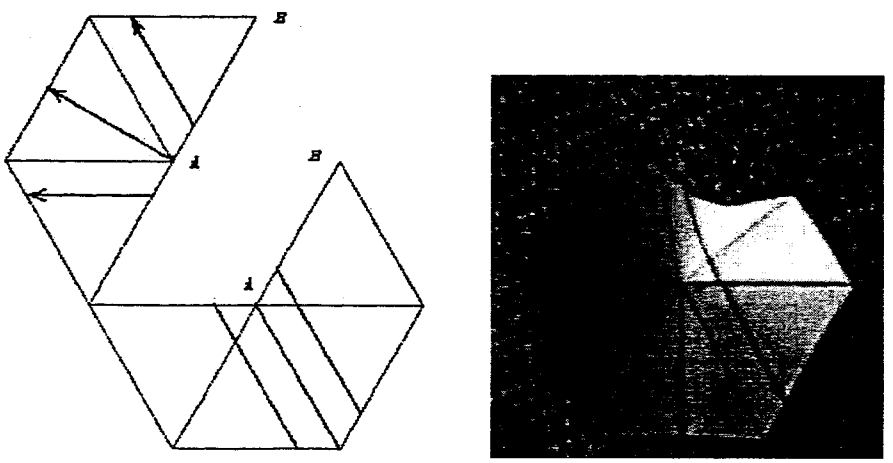
There are five equilateral triangles around an elliptic vertex in an S-manifold. We can take a region around an elliptic vertex and lay it flay by making a cut as in Figure 1a. Note that the lines are straight within any pair of adjacent triangles, although the lines appear to bend at the vertex and across the cut. This is only because we have made a cut and flattened the surface. In the paper model shown in Figure 1b, the lines curve, but only in a direction perpendicular to the surface. In other words, the lines are as straight as possible and bend only as they follow the surface. The two lines that do not pass through the central vertex pass through three adjacent triangles, which would lie flat in the plane, and so are straight in the Euclidean sense. Note that the fact that the third triangle is shared by both lines forces them to intersect. The middle line runs along an edge of a triangle and passes through an elliptic vertex, so it bisects the opposite triangle making two 150° angles (or two-and-a-half triangles). In general, lines passing on either side of an elliptic vertex will turn towards each other.



Figures 1a and 1b. Lines near an elliptic vertex.

Hyperbolic Vertices — seven triangles

There are seven triangles around a hyperbolic vertex. We can lay a region around a hyperbolic vertex flat after making cuts as shown in Figure 2a. The middle line runs along an edge, so it bisects the opposite triangle (and has 210°, or three-and-a-half triangles, on either side of it). The two lines on either side pass through three adjacent triangles, and are straight as in the elliptic case. Note that the third triangles here are separated by another triangle, so lines passing on either side of a hyperbolic vertex turn away from each other.



Figures 2a and 2b. Lines near a hyperbolic vertex.

Paradoxist geometries

We will say that a point P not on a line l is **Euclidean** with respect to l, if there is exactly one line through P that is parallel to l. P is **elliptic** with respect to l, if there are no parallels through P. If there are at least two parallels through P, then it is **hyperbolic**. Furthermore, if P is hyperbolic with respect to l, then it is **finitely hyperbolic**, if there are only finitely many parallels, and it is **regularly hyperbolic**, if there are infinitely many parallels and infinitely many non-parallels. Finally, if there are infinitely many parallels and only finitely many non-parallels, then P is **extremely hyperbolic**, and if all the lines through P are parallel, then P is **completely hyperbolic**.

Smarandache called a geometry **paradoxist** if there are points that are elliptic, Euclidean, finitely hyperbolic, regularly hyperbolic, and completely hyperbolic [1]. We will add extremely hyperbolic to the definition of a paradoxist geometry. We will also say that a geometry is **semi-paradoxist**, if it has Euclidean, elliptic, and regularly hyperbolic points, and if it lacks only finitely hyperbolic points we will call it **almost paradoxist**.

A Semi-Paradoxist Model

This model is constructed by taking a hyperbolic and an elliptic vertex adjacent to each other and surrounding them with Euclidean vertices to form a space that is topologically equivalent to the plane. A part of it is shown in Figures 3a and 3b. Let l be the line through O. With respect to l, we see that the point P is Euclidean. The line through P shown is parallel to l, and any other line through P clearly intersects l, since the region to the right and left is essentially Euclidean.

The point Q is elliptic with respect to l. The line shown intersects l, as would any other line through Q.

The point R is regularly hyperbolic. The lines shown are parallel to l, and these separate the other infinitely many parallels from the infinitely many non-parallels.

This S-manifold can be turned into a Riemannian manifold by smoothing the two curvature singularities. The lines shown in Figures 3a and 3b would stay the same, and only those geodesics passing near the singularities would be affected by the change.

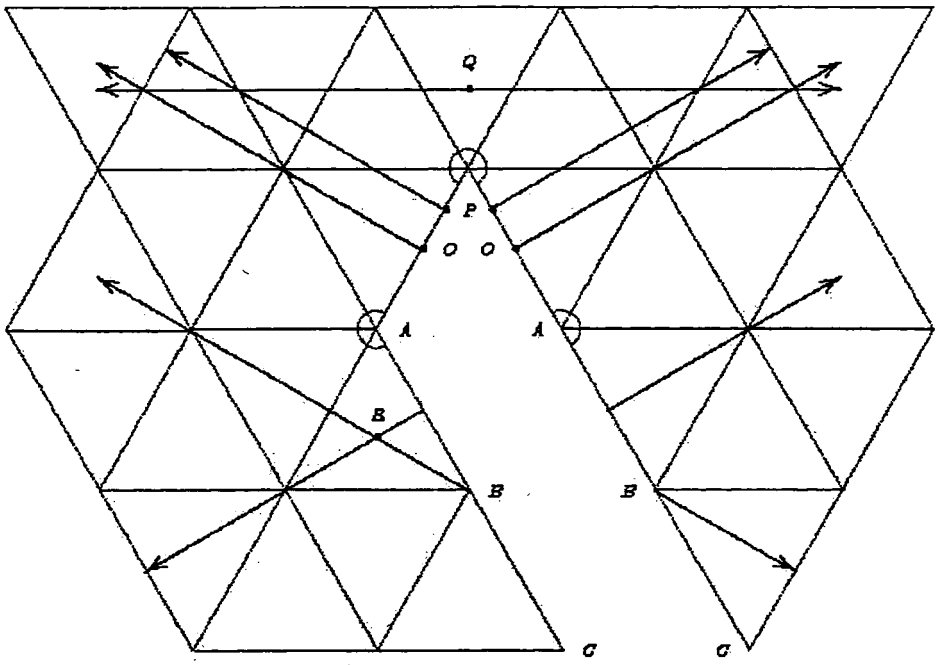


Figure 3a. Lines in the semi-paradoxist model.

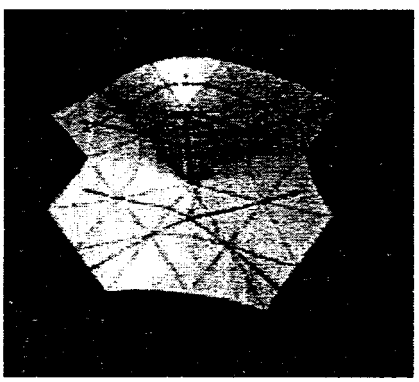


Figure 3b. Lines in the semi-paradoxist model.

An Almost Paradoxist Model

A greater variety in the types of hyperbolic points can be found in an S-manifold with more hyperbolic vertices. This model has at its center an elliptic vertex surrounded by five more elliptic vertices. Five Euclidean vertices then surround these elliptic vertices (see Figures 4a and 4b) to form a cylinder with a cone on top of it. We will call this the silo.

The line *l* runs around the cylinder (it is a circle). With respect to the line *l*, the point *P* is Euclidean, and the point *R* is elliptic.

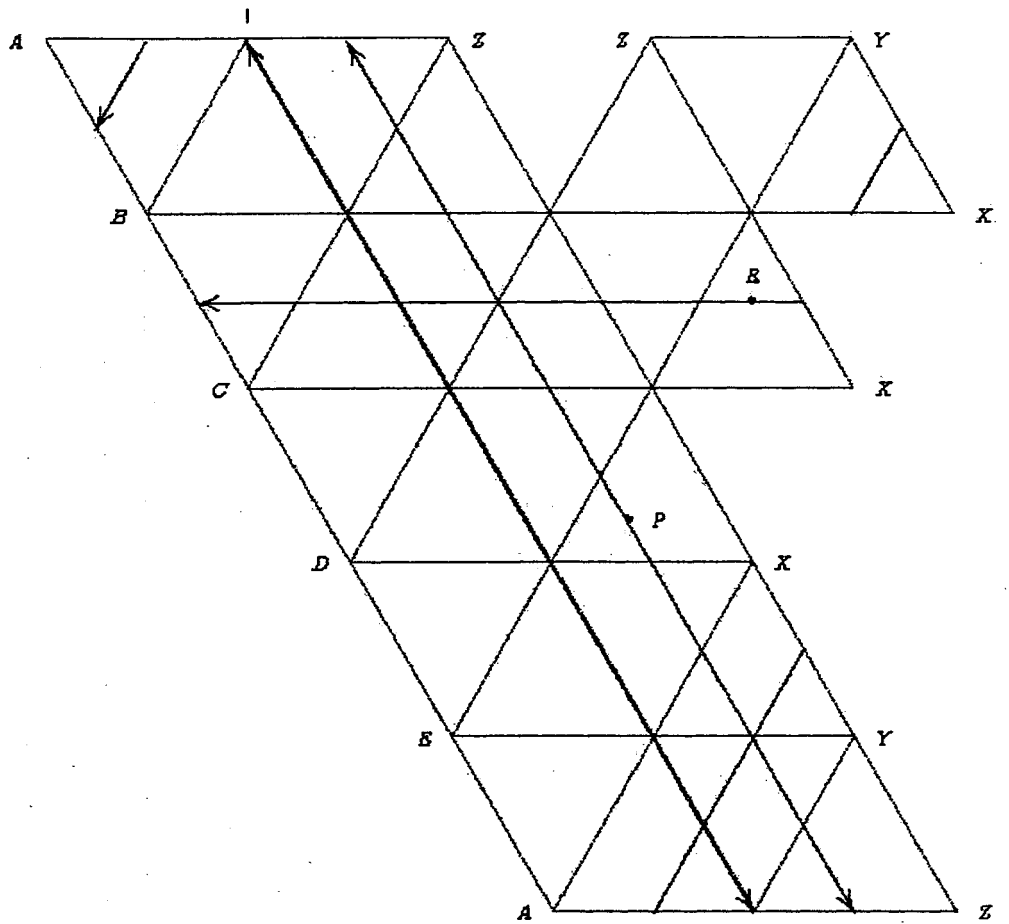
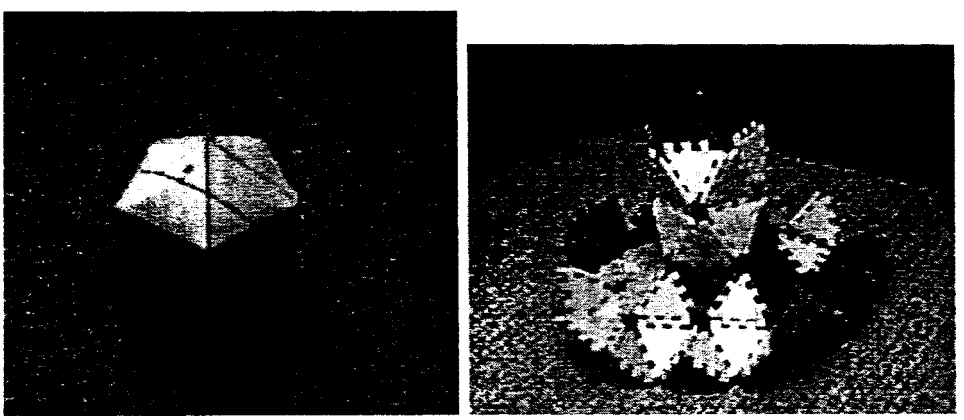


Figure 4a. Lines in the silo of the almost paradoxist model.



Figures 4b and 4c. Lines in the silo of the almost paradoxist model, and the hyperbolic region around silo.

The entire model is topologically equivalent to the plane, and is completed by extending the bottom of the silo with hyperbolic vertices. A model made of *ZAKS blocks* in Figure 4c shows some of the hyperbolic region extending from the bottom of the silo.

Examples of various types of hyperbolic points are shown in Figures 5a and 5b, which shows some of the hyperbolic region and the bottom of the silo. The line l mentioned previously is at the top. With respect to l, the point Q is regularly hyperbolic. The two lines shown are parallel to l, and they separate the parallels from the non-parallels. Out further into the hyperbolic region is the point Q . The line shown passing through Q and the vertex I intersects l. Any line through Q that misses the vertex I will lie outside of the two dotted lines, and these will miss the silo entirely. Since only one line through Q intersects l, it is an extremely hyperbolic point. The nearby point Q is completely hyperbolic. We can see this by noticing that the line through Q and I will follow the dotted line to the left and miss the silo. All the lines through Q to the left of this will also miss the silo. Any line to the right will miss the vertex I, and will run just to the right of the line through Q and I until it misses the vertex F and turns to the right. These lines will also miss the silo.

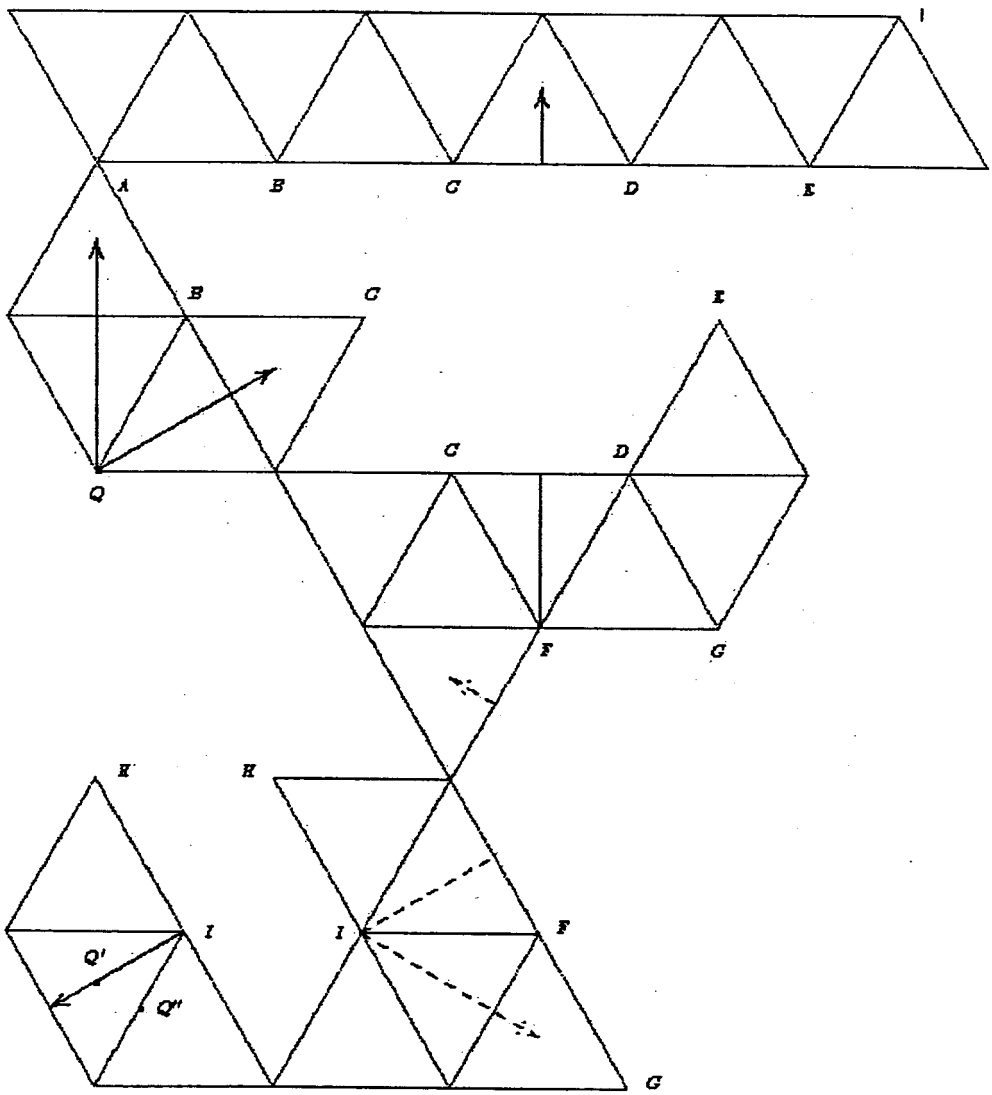


Figure 5a. Lines in the hyperbolic region near the silo in the almost paradoxist model.

Figure 5b. Lines in the hyperbolic region near the silo in the almost paradoxist model.

Since this model has elliptic, Euclidean, regularly hyperbolic, extremely hyperbolic, and completely hyperbolic points, it is almost paradoxist. Note that it follows from the existence of extremely and completely hyperbolic points that there are pairs of points that do not lie on a single line. This model is connected, however, and there is always a finite sequence of line segments that connect any particular pair of points.

Final Remarks

It is relatively easy to construct an S-manifold that is almost paradoxist. The most interesting prospect, however, is the possibility of an S-manifold with a finitely hyperbolic point. Intuition strongly suggests that a finitely hyperbolic point could only exist in a discrete space and not in a continuous space like an S-manifold. A peculiar property of lines in an S-manifold, however, is that a line that passes through a hyperbolic vertex is isolated from lines that are nearby (see Figure 2a). This ability to isolate lines suggests that it may be possible to construct an S-manifold with a finitely hyperbolic point.

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Acknowledgements

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Thanks also to Ken Sullins for introducing me to ZAKS blocks and letting me play with his. Ohio Art's ZAKS blocks were very important in the development and presentation of these models.

ENGINEERING A VISUAL FIELD

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Abstract: *Of the branches of mathematics, geometry has, from the earliest Hellenic period, been given a curious position that straddles empirical and exact science. Its standing as an empirical and approximate science stems from the practical pursuits of artistic drafting, land surveying and measuring in general. From the prominence of visual applications, such as figures and constructions in the twentieth century Einstein's General Theory of Relativity holds that the geometry of space-time is dependent upon physical quantities. On the other hand, earlier on in history, the symmetry and perfect regularity of certain geometric figures were taken as representative of a higher order knowledge than that afforded by sense experience. Concerns with figures and constructions, instead of with numbers and computations, rendered geometry amenable to axiomatic formulation and syllogistic deduction, establishing a paradigm of demonstrative visual and intuitive knowledge that has spanned two millennia.*

In geometry and as followed in geometrical art there remains a connection that distinguishes between the unboundedness of spaces as a property of its extent, and special cases of infinite measure over which distance would be taken is dependent upon particular curvature of lines and spaces. The curvature of a surface could be defined in terms only of properties dependent solely on the surface itself as being intrinsic. On the empirical side, Euclidean and non-Euclidean geometries particularly Riemann's approach effected the understanding of the relationship between geometry and space, in that it stated the question whether space is curved or not. Gauss never published his revolutionary ideas on non-Euclidean geometry, and Bolyai and Lobachevsky are usually credited for their independent discovery of hyperbolic geometry. Hyperbolic geometry is often called Lobachevskian geometry, perhaps because Lobachevsky's work went deeper than Bolyai's. However, in the decades that followed these discoveries Lobachevsky's work met with rather vicious attacks. The decisive figure in the acceptance of non-Euclidean geometry was Beltrami. In 1868, he discovered that hyperbolic geometry could be given a concrete interpretation, via differential geometry. For most purposes, differential geometry is the study of curved surfaces by way of ideas from calculus. Geometries had thus played a part in the emergence and articulation of relativity theory, especially differential geometry. Within the range of mathematical properties these principles could be expressed. Philosophically, geometries stress the hypothetical nature of axiomatizing, contrasting a usual view of mathematical theories as true in some unclear sense. Steadily over the last hundred years the honor of visual reasoning in mathematics has been dishonored. Although the great mathematicians have been oblivious to these fashions the geometer in art has picked up the gauntlet on behalf of geometry. So, metageometry is intended to be in line with the hypothetical character of metaphysics.

Geometric axioms are neither synthetic a priori nor empirical. They are more properly understood as definitions. Thus when one set of axioms is preferred over another the selection is a matter of convention. Poincare's philosophy of science was formed by his approach to mathematics which was broadly geometric. It is governed by the criteria of simplicity of expression rather than by which geometry is ultimately correct. A sketch of Kant's theory of knowledge that defined the existence of mathematical truths a central pillar to his philosophy. In particular, he rests support on the truths of Euclidean

geometry. His inability to realize at that time the existence of any other geometry convinced him that it was the only one. Thereby, the truths demonstrated by Euclidean systems and the existence of a priori synthetic propositions were a guarantee. The discovery of non-Euclidean geometry opened other variables for Kant's arguments. That Euclidean geometry is used to describe the motion of bodies in space, it makes no sense to ask if physical space is really Euclidean. Discovery in mathematics is similar to the discovery in the physical sciences whereas the former is a construction of the human mind. The latter must be considered as an order of nature that is independent of mind. Newton became disenchanted with his original version of calculus and that of Leibniz and around 1680 had proceeded to develop a third version of calculus based on geometry. This geometric calculus is the mathematical engine behind Newton's *Principia.*

Conventionalism as geometrical and mathematical truths are created by our choices, not dictated by or imposed on us by scientific theory. The idea that geometrical truth is truth we create by the understanding of certain conventions in the discovery of non-Euclidean geometries. Subsequent to this discovery, Euclidean geometries had been considered as a paradigm of a priori knowledge. The further discovery of alternative systems of geometry are consistent with making Euclidean geometry seem dismissed without interfering with rationality. Whether we utilize the Euclidean system or non-Euclidean system seems to be a matter of choice founded on pragmatic considerations such as simplicity and convenience.

The Euclidean, Lobachevsky-Bolyai-Gauss, and Reimannian geometries are united in the same space, by the *Smarandache Geometries*, 1969. These geometries are, therefore partially Euclidean and partially Non-Euclidean. The geometries in their importance unite and generalize all together and separate them as well. Hilbert's relations of incidence, betweenness, and congruence are made clearer through the negations of Smarandache's Anti-Geometry. Florentin Smarandache's geometries fall under the following categories: Paradoxist Geometry, Non-Geometry, Counter-Projective Geometry, and Anti-Geometry.

Science provides a fruitful way of expressing the relationships between types or sets of sensations, enabling reliable predictions to be offered. These sensations of sets of data reflect the world that causes them or causal determination; as a limited objectivity of science that derives from this fact, but science does not suppose to determine the nature of that underlying world. It is the underlying structure found through geometry that has driven the world of geometers to artistic expressions. Geometrical art can through conventions and choices which are determinable by rule may appear to be empirical, but are in fact postulates that geometers have chosen to select as implicit definitions. The choice to select a particular curve to represent a finite set of points requires a judgment as to that which is simpler. There are theories which can be drawn that lead to postulate underlying entities or structures. These abstract entities or models may seem explanatory, but strictly speaking are no more than visual devices useful for calculation.

Abstract entities, are sometimes collected under universal categories, that include mathematical objects, such as numbers, sets, and geometrical figures, propositions, and relations. Abstracta, are stated to be abstracted from particulars. The abstract square or triangle have only the properties common to all squares or triangles, and none peculiar to any particular square or triangle; that they have not particular color, size, or specific type whereby they may be used for an artistic purpose. Abstracta are admitted to an ontology by Quine's criterion if they must exist in order to make the mechanics of the structure to be real and true. Properties and relations may be needed to account for resemblance among particulars, such as the blueness shared amongst all blue things.

Concrete intuition and understanding is a major role in the appreciation of geometry as intersections both in art and science. This bares great value not only to the participating geometer artists

but to the scholars for their research. In the presentation of geometry, we can bridge visual intuitive aspects with visual imagination. In this statement, I have outlined for geometry and art without strict definitions of concepts or with any actual computations. Thus, the presentation of geometry as a brushstroke to approach visual intuition should give a much broader range of appreciation to mathematics.

Clifford Singer, 2001 ©

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On Smarandache's Podaire Theorem

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Let A', B', C' be the feet of the altitudes of an acute-angled triangle ABC ($A' \in BC$, $B' \in AC$, $C' \in AB$). Let a', b', b' denote the sides of the podaire triangle $A'B'C'$. Smarandache's Podaire theorem [2] (see [1]) states that

$$\sum a'b' \leq \frac{1}{4}\sum a^2 \tag{1}$$

where a, b, c are the sides of the triangle ABC. Our aim is to improve (1) in the following form:

$$\sum a'b' \leq \frac{1}{3}\left(\sum a'\right)^2 \leq \frac{1}{12}\left(\sum a\right)^2 \leq \frac{1}{4}\sum a^2. \tag{2}$$

First we need the following auxiliary proposition.

Lemma. *Let p and p' denote the semi-perimeters of triangles ABC and $A'B'C'$, respectively. Then*

$$p' \leq \frac{p}{2}. \tag{3}$$

Proof. Since $AC' = b\cos A$, $AB' = c\cos A$, we get

$$C'B' = AB'^2 + AC'^2 - 2AB' \cdot AC' \cdot \cos A = a^2\cos^2 A,$$

so $C'B' = a\cos A$. Similarly one obtains

$$A'C' = b\cos B, \quad A'B' = c\cos C.$$

Therefore

$$p' = \frac{1}{2}\sum A'B' = \frac{1}{2}\sum a\cos A = \frac{R}{2}\sum \sin 2A = 2R\sin A\sin B\sin C$$

(where R is the radius of the circumcircle). By $a = 2R\sin A$, etc. one has

$$p' = 2R\prod \frac{a}{2R} = \frac{S}{R},$$

where $S = area(ABC)$. By $p = \dfrac{S}{r}$ (r = radius of the incircle) we obtain

$$p' = \frac{r}{R}p. \tag{4}$$

Now, Euler's inequality $2r \leq R$ gives relation (3).

For the proof of (2) we shall apply the standard algebraic inequalities

$$3(xy + xz + yz) \leq (x + y + z)^2 \leq 3(x^2 + y^2 + z^2).$$

Now, the proof of (2) runs as follows:

$$\sum a'b' \leq \frac{1}{3}\left(\sum a'\right)^2 = \frac{1}{3}(2p')^2 \leq \frac{1}{3}p^2 = \frac{1}{3}\frac{\left(\sum a\right)^2}{4} \leq \frac{1}{4}\sum a^2.$$

Remark. Other properties of the podaire triangle are included in a recent paper of the author ([4]), as well as in his monograph [3].

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On a dual of the Pseudo-Smarandache function

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1 Introduction

In paper [3] we have defined certain generalizations and extensions of the Smarandache function. Let $f : \mathbb{N}^* \to \mathbb{N}^*$ be an arithmetic function with the following property: for each $n \in \mathbb{N}^*$ there exists at least a $k \in \mathbb{N}^*$ such that $n|f(k)$. Let

$$F_f : \mathbb{N}^* \to \mathbb{N}^* \text{ defined by } F_f(n) = \min\{k \in \mathbb{N}^* : \ n|f(k)\}. \tag{1}$$

This function generalizes many particular functions. For $f(k) = k!$ one gets the Smarandache function, while for $f(k) = \frac{k(k+1)}{2}$ one has the Pseudo-Smarandache function Z (see [1], [4-5]). In the above paper [3] we have defined also dual arithmetic functions as follows: Let $g : \mathbb{N}^* \to \mathbb{N}^*$ be a function having the property that for each $n \geq 1$ there exists at least a $k \geq 1$ such that $g(k)|n$.

Let

$$G_g(n) = \max\{k \in \mathbb{N}^* : \ g(k)|n\}. \tag{2}$$

For $g(k) = k!$ we obtain a dual of the Smarandache function. This particular function, denoted by us as S_* has been studied in the above paper. By putting $g(k) = \frac{k(k+1)}{2}$ one obtains a dual of the Pseudo-Smarandache function. Let us denote this function, by analogy by Z_*. Our aim is to study certain elementary properties of this arithmetic function.

2 The dual of yhe Pseudo-Smarandache function

Let

$$Z_*(n) = \max\left\{m \in \mathbb{N}^* : \frac{m(m+1)}{2} | n\right\}. \tag{3}$$

Recall that

$$Z(n) = \min\left\{k \in \mathbb{N}^* : n | \frac{k(k+1)}{2}\right\}. \tag{4}$$

First remark that

$$Z_*(1) = 1 \quad \text{and} \quad Z_*(p) = \begin{cases} 2, & p = 3 \\ 1, & p \neq 3 \end{cases} \tag{5}$$

where p is an arbitrary prime. Indeed, $\frac{2 \cdot 3}{2} = 3|3$ but $\frac{m(m+1)}{2}|p$ for $p \neq 3$ is possible only for $m = 1$. More generally, let $s \geq 1$ be an integer, and p a prime. Then:

Proposition 1.

$$Z_*(p^s) = \begin{cases} 2, & p = 3 \\ 1, & p \neq 3 \end{cases} \tag{6}$$

Proof. Let $\frac{m(m+1)}{2}|p^s$. If $m = 2M$ then $M(2M+1)|p^s$ is impossible for $M > 1$ since M and $2M+1$ are relatively prime. For $M = 1$ one has $m = 2$ and $3|p^s$ only if $p = 3$. For $m = 2M-1$ we get $(2M-1)M|p^k$, where for $M > 1$ we have $(M, 2M-1) = 1$ as above, while for $M = 1$ we have $m = 1$.

The function Z_* can take large values too, since remark that for e.g. $n \equiv 0(\text{mod}6)$ we have $\frac{3 \cdot 4}{2} = 6|n$, so $Z_*(n) \geq 3$. More generally, let a be a given positive integer and n selected such that $n \equiv 0(\text{mod}a(2a+1))$. Then

$$Z_*(n) \geq 2a. \tag{7}$$

Indeed, $\frac{2a(2a+1)}{2} = a(2a+1)|n$ implies $Z_*(n) \geq 2a$.

A similar situation is in

Proposition 2. Let q be a prime such that $p = 2q-1$ is a prime, too. Then

$$Z_*(pq) = p. \tag{8}$$

Proof. $\frac{p(p+1)}{2} = pq$ so clearly $Z_*(pq) = p$.

Remark. Examples are $Z_*(5\cdot 3) = 5$, $Z_*(13\cdot 7) = 13$, etc. It is a difficult open problem that for infinitely many q, the number p is prime, too (see e.g. [2]).

Proposition 3. For all $n \geq 1$ one has

$$1 \leq Z_*(n) \leq Z(n). \tag{9}$$

Proof. By (3) and (4) we can write $\frac{m(m+1)}{2} | n | \frac{k(k+1)}{2}$, therefore $m(m+1) | k(k+1)$. If $m > k$ then clearly $m(m+1) > k(k+1)$, a contradiction.

Corollary. One has the following limits:

$$\underline{\lim}_{n\to\infty} \frac{Z_*(n)}{Z(n)} = 0, \quad \overline{\lim}_{n\to\infty} \frac{Z_*(n)}{Z(n)} = 1. \tag{10}$$

Proof. Put $n = p$ (prime) in the first relation. The first result follows by (6) for $s = 1$ and the well-known fact that $Z(p) = p$. Then put $n = \frac{a(a+1)}{2}$, when $\frac{Z_*(n)}{Z(n)} = 1$ and let $a \to \infty$.

As we have seen,

$$Z\left(\frac{a(a+1)}{2}\right) = Z_*\left(\frac{a(a+1)}{2}\right) = a.$$

Indeed, $\frac{a(a+1)}{2} \Big| \frac{k(k+1)}{2}$ is true for $k = a$ and is not true for any $k < a$. In the same manner, $\frac{m(m+1)}{2} \Big| \frac{a(a+1)}{2}$ is valied for $m = a$ but not for any $m > a$. The following problem arises: What are the solutions of the equation $Z(n) = Z_*(n)$?

Proposition 4. All solutions of equation $Z(n) = Z_*(n)$ can be written in the form $n = \frac{r(r+1)}{2}$ $(r \in \mathbb{N}^*)$.

Proof. Let $Z_*(n) = Z(n) = t$. Then $n | \frac{t(t+1)}{2} | n$ so $\frac{t(t+1)}{2} = n$. This gives $t^2 + t - 2n = 0$ or $(2t+1)^2 = 8n + 1$, implying $t = \frac{\sqrt{8n+1} - 1}{2}$, where $8n + 1 = m^2$. Here m must be odd, let $m = 2r + 1$, so $n = \frac{(m-1)(m+1)}{8}$ and $t = \frac{m-1}{2}$. Then $m - 1 = 2r$, $m + 1 = 2(r+1)$ and $n = \frac{r(r+1)}{2}$.

Proposition 5. One has the following limits:

$$\lim_{n\to\infty} \sqrt[n]{Z_*(n)} = \lim_{n\to\infty} \sqrt[n]{Z(n)} = 1. \tag{11}$$

Proof. It is known that $Z(n) \leq 2n-1$ with equality only for $n = 2^k$ (see e.g. [5]). Therefore, from (9) we have

$$1 \leq \sqrt[n]{Z_*(n)} \leq \sqrt[n]{Z(n)} \leq \sqrt[n]{2n-1},$$

and by taking $n \to \infty$ since $\sqrt[n]{2n-1} \to 1$, the above simple result follows.

As we have seen in (9), upper bounds for $Z(n)$ give also upper bounds for $Z_*(n)$. E.g. for $n =$ odd, since $Z(n) \leq n-1$, we get also $Z_*(n) \leq n-1$. However, this upper bound is too large. The optimal one is given by:

Proposition 6.

$$Z_*(n) \leq \frac{\sqrt{8n+1}-1}{2} \text{ for all } n. \tag{12}$$

Proof. The definition (3) implies with $Z_*(n) = m$ that $\frac{m(m+1)}{2} | n$, so $\frac{m(m+1)}{2} \leq n$, i.e. $m^2 + m - 2n \leq 0$. Resolving this inequality in the unknown m, easily follows (12). Inequality (12) cannot be improved since for $n = \frac{p(p+1)}{2}$ (thus for infinitely many n) we have equality. Indeed,

$$\left(\sqrt{\frac{8(p+1)p}{2}+1}-1\right)/2 = \left(\sqrt{4p(p+1)+1}-1\right)/2 = [(2p+1)-1]/2 = p.$$

Corollary.

$$\underline{\lim}_{n\to\infty} \frac{Z_*(n)}{\sqrt{n}} = 0, \quad \overline{\lim}_{n\to\infty} \frac{Z_*(n)}{\sqrt{n}} = \sqrt{2}. \tag{13}$$

Proof. While the first limit is trivial (e.g. for $n =$ prime), the second one is a consequence of (12). Indeed, (12) implies $Z_*(n)/\sqrt{n} \leq \sqrt{2}\left(\sqrt{1+\frac{1}{8n}} - \sqrt{\frac{1}{8n}}\right)$, i.e. $\overline{\lim}_{n\to\infty} \frac{Z_*(n)}{\sqrt{n}} \leq \sqrt{2}$. But this upper limit is exact for $n = \frac{p(p+1)}{2}$ $(p \to \infty)$.

Similar and other relations on the functions S and Z can be found in [4-5].

An inequality connecting $S_*(ab)$ with $S_*(a)$ and $S_*(b)$ appears in [3]. A similar result holds for the functions Z and Z_*.

Proposition 7. For all $a, b \geq 1$ one has

$$Z_*(ab) \geq \max\{Z_*(a), Z_*(b)\}, \tag{14}$$

$$Z(ab) \geq \max\{Z(a), Z(b)\} \geq \max\{Z_*(a), Z_*(b)\}. \tag{15}$$

Proof. If $m = Z_*(a)$, then $\frac{m(m+1)}{2} | a$. Since $a|ab$ for all $b \geq 1$, clearly $\frac{m(m+1)}{2} | ab$, implying $Z_*(ab) \geq m = Z_*(a)$. In the same manner, $Z_*(ab) \geq Z_*(b)$, giving (14).

Let now $k = Z(ab)$. Then, by (4) we can write $ab | \frac{k(k+1)}{2}$. By $a|ab$ it results $a | \frac{k(k+1)}{2}$, implying $Z(a) \leq k = Z(ab)$. Analogously, $Z(b) \leq Z(ab)$, which via (9) gives (15).

Corollary. $Z_*(3^s \cdot p) \geq 2$ for any integer $s \geq 1$ and any prime p. (16)

Indeed, by (14), $Z_*(3^s \cdot p) \geq \max\{Z_*(3^s), Z(p)\} = \max\{2, 1\} = 2$, by (6).

We now consider two irrational series.

Proposition 8. The series $\sum_{n=1}^{\infty} \frac{Z_*(n)}{n!}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} Z_*(n)}{n!}$ are irrational.

Proof. For the first series we apply the following irrationality criterion ([6]). Let (v_n) be a sequence of nonnegative integers such that

(i) $v_n < n$ for all large n;

(ii) $v_n < n - 1$ for infinitely many n;

(iii) $v_n > 0$ for infinitely many n.

Then $\sum_{n=1}^{\infty} \frac{v_n}{n!}$ is irrational.

Let $v_n = Z_*(n)$. Then, by (12) $Z_*(n) < n - 1$ follows from $\frac{\sqrt{8n+1} - 1}{2} < n - 1$, i.e. (after some elementary fact, which we omit here) $n > 3$. Since $Z_*(n) \geq 1$, conditions (i)-(iii) are trivially satisfied.

For the second series we will apply a criterion from [7]:

Let $(a_k), (b_k)$ be sequences of positive integers such that

(i) $k | a_1 a_2 \ldots a_k$;

(ii) $\frac{b_{k+1}}{a_{k+1}} < b_k < a_k$ $(k \geq k_0)$. Then $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{b_k}{a_1 a_2 \ldots a_k}$ is irrational.

Let $a_k = k$, $b_k = Z_*(k)$. Then (i) is trivial, while (ii) is $\frac{Z_*(k+1)}{k+1} < Z_*(k) < k$. Here $Z_*(k) < k$ for $k \geq 2$. Further $Z_*(k+1) < (k+1) Z_*(k)$ follows by $1 \leq Z_*(k)$ and $Z_*(k+1) < k + 1$.

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A NEW EQUATION FOR THE LOAD BALANCE SCHEDULING BASED ON THE SMARANDACHE *F*-INFERIOR PART FUNCTION

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Abstract. This article represents an extension of [Tabirca, 2000a]. A new equation for upper bounds is obtained based on the Smarandache f-inferior part function. An example involving upper diagonal matrices is given in order to illustrate that the new equation provide a better computation.

1.INTRODUCTION

Loop imbalance is the most important overhead in many parallel applications. Because loop structures represents the main source of parallelism, the scheduling of parallel loop iterations to processors can determine its decreasing. Among the many method for loop scheduling, the *load balance scheduling* is a recent one and was proposed by Bull [1998] and developed by Freeman *et.al.* [1999, 2000]. Tabirca [2000] studied this method and proposed an equation for the case when the work is distributed to all the processors.

Consider that there are p processors denoted in the following by $P_1, P_2, \ldots, P_p$ and a single parallel loop (see Figure 1.).

```
do parallel i=1,n
        call loop_body(i);
end do
```

Figure 1. Single Parallel Loop

We also assume that the work of the routine loop_body(i) can be evaluated and is given by the function $w: N \to R$, where $w(i) = w_i$ represents the number of routine's operations or its running time (assume that $w(0)=0$). The total amount of work for the parallel loop is $\sum_{i=1}^{n} w(i)$. The efficient loop-scheduling algorithm distributes equally this total amount of

work on processors such that a processor receives a quantity of work equal to $\frac{1}{p}\cdot\sum_{i=1}^{n} w(i)$.

Let l_j and h_j be the lower and upper bounds, $j=1,2,...,p$, such that processor j executes all the iterations between l_j and h_j. These bounds are found distributing equally the work on processors by using

$$\sum_{i=l_j}^{h_j} w(i) \approx \frac{1}{p}\cdot\sum_{i=1}^{n} w(i)\ (\forall j=1,2,...,p). \tag{1}$$

Moreover, they satisfy the following equations

$$l_1 = 1. \tag{2.a}$$

$$\text{if we know } l_j, \text{ then } h_j \text{ is given by } \sum_{i=l_j}^{h_j} w(i) \approx \frac{1}{p}\cdot\sum_{i=1}^{n} w(i) = \overline{W}\ . \tag{2.b}$$

$$l_{j+1} = h_j + 1. \tag{2.c}$$

Suppose that Equation (2.b) is computed by a less approximation. This means that if we have the value l_j, then we find h_j as follows:

$$h_j = h \Leftrightarrow \sum_{i=l_j}^{h} w(i) \le \overline{W} < \sum_{i=l_j}^{h+1} w(i)\ . \tag{3}$$

The Smarandache f-inferior part function represents a generalisation of the inferior part function $[,]: R \to Z$, $[x]=k \Leftrightarrow k \le x < k+1$. If $f: Z \to R$ is a strict increasing function that satisfies $\lim_{n\to-\infty} f(n) = -\infty$ and $\lim_{n\to\infty} f(n) = \infty$, then the Smarandache f-inferior part function denoted by $f_{[]}: R \to Z$ is defined by [see www.gallup.unm.edu/~smarandache]

$$f_{[]}(x) = k \Leftrightarrow f(k) \le x < f(k+1). \tag{4}$$

Tabirca [2000a] presented some Smarandache f-inferior part functions for which $f(k) = \sum_{i=1}^{k} i^a$. They are presented in the following:

$$f(k) = \sum_{i=1}^{k} i \ \Rightarrow\ f_{[]}(x) = \left[\frac{-1+\sqrt{1+8\cdot x}}{2}\right] \forall x \ge 0. \tag{5}$$

$$f(k) = \sum_{i=1}^{k} i^2 \ \Rightarrow\ f_{[]}(x) = [r(x)]\ \forall x \ge 0, \tag{6}$$

where $r(x) = -\frac{1}{2} + \sqrt[3]{\frac{3\cdot x}{2} - \sqrt{\left(\frac{3\cdot x}{2}\right)^2 + \frac{1}{1728}}} + \sqrt[3]{\frac{3\cdot x}{2} + \sqrt{\left(\frac{3\cdot x}{2}\right)^2 + \frac{1}{1728}}}$.

Tabirca [2000] also proposed an equation for the upper bounds of the load balance scheduling method based on the Smarandache f-inferior part function. If the work w satisfies certain conditions [Tabirca, 2000], then the upper bounds are given by

$$h_j^{(1)} = f_{[]}\left(j \cdot \overline{W}\right), j = 1,2,...,p. \tag{7}$$

Moreover, Tabirca [2000a] applied this method to the product between an upper diagonal matrix and a vector. It was proved that the load balance scheduling method offers the lowest running time in comparison with other static scheduling methods [Tabirca, 2000b].

2. A NEW EQUATION FOR THE UPPER BOUNDS

In this section, a new equation for the upper bounds is introduced. Some theoretical considerations about the new equation and Equation (7) are also made. Consider that $f: N \to R$ is defined by $f(k) = \sum_{i=1}^{k} w_i$, $f(0) = 0$. For the work w, we assume the following [Tabirca, 2000]:

A1: $w_j \le \frac{1}{p} \cdot \sum_{i=1}^{n} w_i, j = 1,2,...,n.$

A2: There are equations for the functions $f, f_{[]}$.

Theorem 1. *The upper bounds of the load balance scheduling method are given by*

$$h_j^{(2)} = f_{[]}\left(f(h_{j-1}^{(2)}) + \overline{W}\right), j = 1,2,...,p. \tag{8}$$

Proof. For easiness we denote in the following $h_j = h_j^{(2)}$. Equation (3) gives the upper bounds of the load balance scheduling method. We start from the equation $\sum_{i=l_j}^{h_j} w(i) \le \overline{W} < \sum_{i=l_j}^{h_j+1} w(i)$ and add $f(h_{j-1}) = \sum_{i=1}^{h_{j-1}} w_i$ to all the sides

$$\sum_{i=1}^{h_j} w(i) \le f(h_{j-1}) + \overline{W} < \sum_{i=1}^{h_j+1} w(i).$$

Based on the definition of $f_{[]}$, we find that $h_j = f_{[]}\left(f(h_{j-1}) + \overline{W}\right)$. ♦

The following theorem illustrates how these bounds are.

Theorem 2. $h_j^{(2)} \le h_j^{(1)}, j = 1,2,...,p.$

Proof. Recall that these two upper bounds satisfy

$$\sum_{i=1}^{h_j^{(1)}} w_i \le j \cdot \overline{W} < \sum_{i=1}^{h_j^{(1)}+1} w_i \tag{9.a}$$

$$\sum_{i=l_j^{(2)}}^{h_j^{(2)}} w_i \le \overline{W} < \sum_{i=l_j^{(2)}}^{h_j^{(2)}+1} w_i . \qquad (9.b)$$

All the sums from Equation (9.b.) are added finding

$$\sum_{i=1}^{j} \sum_{k=l_i^{(2)}}^{h_i^{(2)}} w_k \le j \cdot \overline{W} \Leftrightarrow \sum_{i=1}^{h_j^{(2)}} w_i \le j \cdot \overline{W} .$$

Because $h_j^{(1)}$ is the last index satisfying Equation (9.a) we find that $h_j^{(2)} \le h_j^{(1)}$ holds. ♦

Consequence: $f(h_j^{(2)}) \le f(h_j^{(1)}) \le j \cdot \overline{W}, j = 1,2,..., p$.

This consequence obviously comes from the monotony of f and the definition of the bounds.

Now, we have two equations for the upper bounds of the load balance scheduling method. Equation (8) was obtained naturally by starting from the definition of the load balance. It reflects that case when several load balances are performed consecutively. Equation (7) was found by considering the last partial sum that is under $j \cdot \overline{W}$. This option does not consider any load balance such that we expect it to be not quit efficient. Moreover, it is difficult to predict which equation is the best or is better to use it of a given computation. The best practical advice is to apply both of them and to choose the one, which gives the lowest times.

3. COMPUTATIONAL RESULTS

In this section we present an example for the load balance scheduling method. This example deals with the product between an upper diagonal matrix and a vector [Jaja, 1992]. All the computations have been performed on SGI Power Challenge 2000 parallel machine with 16 processors. The dimension of the matrix was n=300.

DO PARALLEL i=1,n
    $y_i = a_{i,1} \cdot x_1$
    DO j=2,i
        $y_i = y_i + a_{i,j} \cdot x_j$
    END DO
END DO

Figure 2. Parallel Computation for the Upper Matrix – Vector Product.

Recall that $a = (a_{i,j})_{i,j=\overline{1,n}} \in M_n(R)$ is upper diagonal if $a_{i,j} = 0, i < j$. The product $y = a \cdot x$ between an upper diagonal matrix $a = (a_{i,j})_{i,j=\overline{1,n}} \in M_n(R)$ and a vector $x \in R^n$ is given by

$$y_i = \sum_{j=1}^{i} a_{i,j} \cdot x_j \ \forall i = 1,2,...n . \qquad (10)$$

The parallel computation of Equation (10) is shown in Figure 2.

The work of iteration i is given by $w(i)=i, i=1,2,...,n$. We have that the total work is $f(n)=\sum_{i=1}^{n} i=\frac{n\cdot(n+1)}{2}$ and $\overline{W}=\frac{n\cdot(n+1)}{2\cdot p}$. The Smarandache f-inferior function is $f_{\square}(x)=\left[\frac{-1+\sqrt{1+8\cdot x}}{2}\right] \forall x \geq 0$. Therefore, the upper bounds of the load balance scheduling method are given by

$$h_j^{(1)}=\left[\frac{-1+\sqrt{1+4\cdot j\cdot\frac{n\cdot(n+1)}{p}}}{2}\right], j=1,2,...,p \quad \text{or} \quad (11)$$

$$h_j^{(2)}=\left[\frac{-1+\sqrt{1+4\cdot h_{j-1}^{(2)}\cdot(h_{j-1}^{(2)}+1)+4\cdot\frac{n\cdot(n+1)}{p}}}{2}\right], j=1,2,...,p. \quad (12)$$

The running times for these two types of upper bounds are presented in Table 1. Figure 3 proves that these two types of bounds for the load balance scheduling are comparable the same.

	P=1	*P=2*	*P=3*	*P=6*	*P=8*
$h_j^{(1)}$	1.847	1.347	0.987	0.750	0.482
$h_j^{(2)}$	1.842	1.258	0.832	0.639	0.412

Table 1. Times of the computation.

4. FINAL CONCLUSSION

An important remark that can be outlined is the Smarandache inferior part function was applied successfully to solve an important scheduling problem. Based on it, two equations for the upper bounds of the load balance scheduling methods have been found. These equations have been used to solve the product between an upper diagonal matrix and vector and the computational times were quite similar. The upper bounds given by the new equation have provided a better computation for this problem.

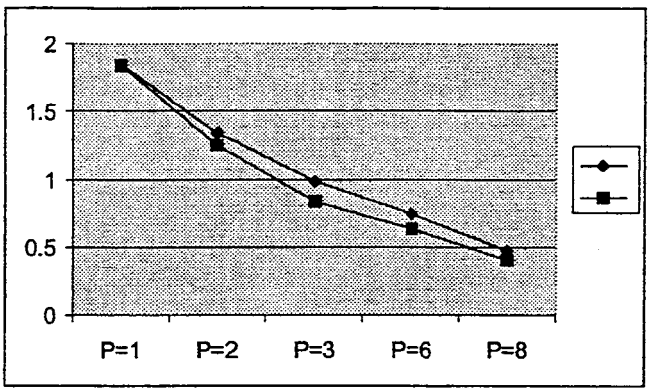

Figure 3. Graphics of the Running Times.

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SOME NEW RESULTS CONCERNING THE SMARANDACHE CEIL FUNCTION

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Abstract: In this article we present two new results concerning the Smarandache Ceil function. The first result proposes an equation for the number of fixed-point number of the Smarandache ceil function. Based on this result we prove that the average of the Smarandache ceil function is $\Theta(n)$.

1. INTRODUCTION

In this section we review briefly the main results that are used in this article. These concern the Smarandache ceil and functions. The Smarandache ceil function of order k [see www.gallup.unm.edu/~smarandache] is denoted by $S_k : N^* \to N$ and has the following definition

$$S_k(n) = \min\{x \in N \mid x^k \vdots n\} (\forall n \in N^*). \tag{1}$$

This was introduced by Smarandache [1993] who proposed many open problems concerning it. Ibstedt [1997, 1999] studied this function both theoretically and computationally. The main properties proposed in [Ibstedt, 1997] are presented in the following

$$(\forall a, b \in N^*)(a,b) = 1 \Rightarrow S_k(a \cdot b) = S_k(a) \cdot S_k(b), \tag{2.a}$$

$$S_k(p_1^{a_1} \cdot \ldots \cdot p_s^{a_s}) = S(p_1^{a_1}) \cdot \ldots \cdot S(p_s^{a_s})\} \text{ and} \tag{2.b}$$

$$S_k(p^a) = p^{\left\lceil \frac{a}{k} \right\rceil}. \tag{2.b}$$

Therefore, if $n = p_1^{a_1} \cdot \ldots \cdot p_s^{a_s}$ is the prime number decomposition of n, then the equation of this function is given by

$$S_k(p_1^{a_1} \cdot \ldots \cdot p_s^{a_s}) = p_1^{\left\lceil \frac{a_1}{k} \right\rceil} \cdot \ldots \cdot p_s^{\left\lceil \frac{a_s}{k} \right\rceil}. \tag{3}$$

Based on these properties, Ibstedt proposed the following results

$$S_{k+1}(n) \vdots S_k(n) \; \forall n > 1 \tag{4}$$

$$n = p_1 \cdot \ldots \cdot p_s \Rightarrow S_2(n) = n. \tag{5}$$

Table 1 shows the values of the Smarandache ceil function of order 2 for $n<25$.

n	$S_2(n)$	N	$S_2(n)$	N	$S_2(n)$	N	$S_2(n)$	n	$S_2(n)$
1	1	6	6	11	11	16	4	21	21
2	2	7	7	12	6	17	17	22	22
3	3	8	4	13	13	18	6	23	23
4	2	9	3	14	14	19	19	24	12
5	5	10	10	15	15	20	10	25	5

Table 1. The Smarandache ceil function.

The Mobius function $\mu : N \to Z$ is defined as follows

$$\mu(1) = 1 \tag{6.a}$$

$$\mu(n) = (-1)^s \text{ if } n = p_1 \cdot \ldots \cdot p_s \tag{6.b}$$

$$\mu(n) = 0 \text{ otherwise}. \tag{6.c}$$

This is an important function both in Number Theory and Combinatorics because gives two inversion equations. The first Mobius inversion formula [Chandrasekharan, 1970] is

$$g(n) = \sum_{d|n} f(d) \Leftrightarrow f(n) = \sum_{d|n} \mu(d) \cdot g(\frac{n}{d}) \tag{7.a}$$

while the second Mobius formula is

$$g(x) = \sum_{n \le x} f(\frac{x}{n}) \Leftrightarrow f(x) = \sum_{n \le x} \mu(n) \cdot g(\frac{x}{n}). \tag{7.b}$$

There are several equations concerning series involving the Mobius function [Apostol, 1976]. Among them an important series is

$$\sum_{n>0} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2} \tag{8.a}$$

that has the following asymptotic form

$$\sum_{0<n\le x} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2} + O(\frac{1}{x}). \tag{8.b}$$

2. THE ASYMPTOTIC DENSITY OF FIXED POINTS

In this section we present an equation for the asymptotic density of the function S_k's fixed points. The main result presented can also be found in [Keng, 1981] but we give it a detailed proof. We start by remarking that the function S_2 has quit many points. For example, there are 16 fixed points for the first 25 numbers.

Let $q(x)$ be the number of the fixed points less than x: $q(x)=\#\{n \le x : S_k(n)=n\}$. We say that the fixed points have the asymptotic density equal to a if $\lim_{x\to\infty}\frac{q(x)}{x}=a$.

Ibstedt [1997] found that if n is a square free number then it is a fixed point for S_2. Actually, the result holds for any Smarandache ceil function.

Proposition 1. $n=p_1\cdot\ldots\cdot p_s \Leftrightarrow S_k(n)=n$.

Proof Let $n=p_1^{a_1}\cdot\ldots\cdot p_s^{a_s}$ be the prime number decomposition of n. The following equivalence gives the proof:

$$S_k(n)=n \Leftrightarrow p_1^{a_1}\cdot\ldots\cdot p_s^{a_s}=p_1^{\left\lceil\frac{a_1}{k}\right\rceil}\cdot\ldots\cdot p_s^{\left\lceil\frac{a_s}{k}\right\rceil}\Leftrightarrow$$

$$\left\lceil\frac{a_i}{k}\right\rceil=a_i, i=1,2,\ldots,s \Leftrightarrow a_i=1, i=1,2,\ldots,s \Leftrightarrow n=p_1\cdot\ldots\cdot p_s.$$

Therefore, n is a square free number. ♦

Proposition 2. $(\forall n\in N)(\exists! d\mid n)\frac{n}{d^2}$ is square free. (9)

Proof. Firstly, we prove that there is such as divisor. If $n=p_1^{a_1}\cdot\ldots\cdot p_s^{a_s}$ the prime number decomposition, then $d=p_1^{\left\lfloor\frac{a_1}{2}\right\rfloor}\cdot\ldots\cdot p_s^{\left\lfloor\frac{a_s}{2}\right\rfloor}$ satisfies $\frac{n}{d^2}$ is square free. Actually, $\frac{n}{d^2}$ is the product of all prime numbers that have odd power in the prime number decomposition of n. Now, we prove that d is unique. Assume that there are distinct divisors such that $\frac{n}{d_1^2},\frac{n}{d_2^2}$ are square free. We can write this as follows $n=d_1^2\cdot p_1\cdot\ldots\cdot p_s=d_2^2\cdot q_1\cdot\ldots\cdot q_r$. Let p be a prime number that does not appear in the both sites $p_1,\ldots,p_s$ and $q_1,\ldots,q_r$ (choose that it is in the first). p

should also appear in the prime number decomposition of d_2^2. Therefore, we find that the power of p is even for the right hand side and odd for the left hand side. ♦

Proposition 3. $\{0 < n \le x\} = \bigcup_{d=1}^{\lfloor\sqrt{x}\rfloor} d^2 \cdot \{i \le \frac{x}{d^2} : \text{i is square free}\}$. (10)

Proof. It is enough to prove this equation just for natural number. Consider n>1 a natural number. Equation (10) becomes

$$\{1,2,...,n\} = \bigcup_{d=1}^{\lfloor\sqrt{n}\rfloor} d^2 \cdot \{i \le \frac{n}{d^2} : \text{i is square free}\}. \quad (11)$$

The inclusion $\{1,2,...,n\} \supseteq \bigcup_{d=1}^{\lfloor\sqrt{n}\rfloor} d^2 \cdot \{i \le \frac{n}{d^2} : \text{i is square free}\}$ is obviously true. A number $i \le n$ can be written uniquely as $i = d^2 \cdot d_1$ where $d \le \lfloor\sqrt{i}\rfloor \le \lfloor\sqrt{n}\rfloor$ and d_1 is square free. We find that it belongs to $d^2 \cdot \{i \le \frac{n}{d^2} : \text{i is square free}\}$, thus Equation (10) holds. ♦

Consequence: Taking the number of elements in Equation (10) we find

$$\lfloor x \rfloor = \sum_{i=1}^{\lfloor\sqrt{x}\rfloor} q\left(\frac{x}{i^2}\right) \forall\, x > 0. \quad (12)$$

Based on this result and on Equations (7-8) the following theorem is found.

Theorem 4. [Keng] $q(x) = \frac{6}{\pi^2} \cdot x + O(\sqrt{x})$ (13)

Proof. For $x = y^2$, Equation (12) gives $\lfloor y^2 \rfloor = \sum_{i=1}^{\lfloor y\rfloor} q\left(\left(\frac{y}{i}\right)^2\right)$. The second Mobius inversion formula gives

$$q(y^2) = \sum_{i=1}^{\lfloor y\rfloor} \mu(i) \cdot \left\lfloor \frac{y^2}{i^2} \right\rfloor. \quad (14)$$

Equation (14) is transformed based on Equation (8.b) as follows

$$q(y^2) = \sum_{i=1}^{\lfloor y\rfloor} \mu(i) \cdot \left\lfloor \frac{y^2}{i^2} \right\rfloor = \sum_{i=1}^{\lfloor y\rfloor} \mu(i) \cdot \left(\frac{y^2}{i^2} - \left\{ \frac{y^2}{i^2} \right\} \right) = \sum_{i=1}^{\lfloor y\rfloor} \mu(i) \cdot \frac{y^2}{i^2} - \sum_{i=1}^{\lfloor y\rfloor} \mu(i) \cdot \left\{ \frac{y^2}{i^2} \right\}$$

$$= y^2 \cdot \sum_{i=1}^{\lfloor y\rfloor} \frac{\mu(i)}{i^2} + O(y) = \frac{6}{\pi^2} \cdot y^2 + y^2 \cdot O(\frac{1}{y}) + O(y) = \frac{6}{\pi^2} \cdot y^2 + O(y).$$

Equation (13) is obtained from the last one by substituting $x = y^2$. ♦

Consequence: $\lim_{x\to\infty}\frac{q(x)}{x}=\frac{6}{\pi^2}$. (15)

Equation (15) gives that the asymptotic density for the fixed points of the Smarandache ceil function is $\frac{6}{\pi^2}$. Because $\frac{6}{\pi^2}=0.607927...$, we find that more than 60% of points are fixed points. Equation (15) also produces an algorithm for approximating π that is described in the following.

Step 1. Find the number of fixed points for the Smarandache ceil function S_2.

Step 2. Find the approximation of π by using $\pi \approx \sqrt{\frac{6\cdot x}{q(x)}}$.

3. THE AVERAGE OF THE SMARANDACHE CEIL FUNCTION

In this section we study the Θ complexity of the average of the Smarandache ceil function. Let $\overline{S}_k(n)=\frac{\sum_{i=1}^{n}S_k(i)}{n}$ be the average of the Smarandache ceil function. Recall that $f(n)=\Theta(g(n))$ if $(\exists C_1,C_2>0)(\forall n>n_0)C_1\cdot g(n)\le f(n)\le C_2\cdot g(n)$ [Bach, 1996].

Theorem 5. The Θ-complexity of the average $\overline{S}_k(n)$ is given by

$$\overline{S}_k(n)=\Theta(n). \tag{16}$$

Proof. This result is obtained from Equation (15). One inequality is obviously obtained as follows $\overline{S}_k(n)=\frac{\sum_{i=1}^{n}S_k(i)}{n}\le\frac{\sum_{i=1}^{n}i}{n}=\frac{n+1}{2}$.

Because $\lim_{x\to\infty}\frac{q(x)}{x}=\frac{6}{\pi^2}>\frac{1}{2}$, we find that $q(x)>\frac{x}{2},\forall x>x_0$. Therefore, there are at least 50% fixed points. Consider that $i_1=1,i_2=2,...,i_{q(n)}$ are the fixed points less than n for the Smarandache ceil function. These obviously satisfy $i_j\ge j, j=1,2,...,q(n)$.

Now, we keep in the average only the fixed points

$$\overline{S}_k(n)=\frac{\sum_{i=1}^{n}S_k(i)}{n}\geq\frac{\sum_{j=1}^{q(n)}S_k(i_j)}{n}=\frac{\sum_{j=1}^{q(n)}i_j}{n}\geq\frac{\sum_{j=1}^{q(n)}j}{n}=\frac{q(n)\cdot(q(n)+1)}{2\cdot n}.$$

Because $q(n)>\frac{n}{2}$, we find that $\overline{S}_k(n)\geq\frac{\frac{n}{2}\cdot(\frac{n}{2}+1)}{2\cdot n}=\frac{n}{8}+\frac{1}{4}$ for each $n>x_0$.

Therefore, the average function satisfies

$$\frac{n}{8}+\frac{1}{4}\leq\overline{S}_k(n)\leq\frac{n}{2}+\frac{1}{2}\forall n>x_0 \tag{16}$$

that gives the Θ-complexity is $\overline{S}_k(n)=\Theta(n)$. ♦

This Θ-complexity complexity gives that the average of the Smarandache ceil function is linear. Unfortunately, we have not been able to find more details about the average function behavior.

What is ideally to find is $C\in\left(\frac{1}{8},\frac{1}{2}\right)$ such that

$$\overline{S}_k(n)=C\cdot n+O(n^{1-\varepsilon}). \tag{17}$$

From Equation (17) we find the constant C is $C=\lim_{n\to\infty}\frac{\overline{S}_k(n)}{n}=\lim_{n\to\infty}\frac{\sum_{i=1}^{n}S_k(i)}{n^2}$.

Example. For the Smarandache ceil function S_2 we have found by using a simple computation that $\frac{\sum_{i=1}^{n}S_2(i)}{n^2}\approx 0.3654...$ and $\sqrt{n}\cdot\left[\frac{\sum_{i=1}^{n}S_2(i)}{n^2}-0.3654\right]\approx 0.038...$, which give the $\overline{S}_2(n)\approx 0.3654\cdot n+0.038\cdot\sqrt{n}$

This example makes us to believe that the following conjecture holds.

Conjecture: There is a constant $C\in\left(\frac{1}{8},\frac{1}{2}\right)$ such that $\overline{S}_k(n)=C\cdot n+O(n^{1-\frac{1}{k}})$. (18)

4. CONCLUSSIONS

This article has presented two important results concerning the Smarandache ceil function. We firstly have established that the asymptotic density of fixed points is $\frac{6}{\pi^2}$. Based on this we have found the average function of the Smarandache ceil function behaves linearly. Based on a simple computation the following Equation (18) has been conjectured.

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BOUNDING THE SMARANDACHE FUNCTION

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Let $S(n)$, for $n \in \mathbb{N}^+$ denote the Smarandache function, then $S(n)$ is defined as the smallest $m \in \mathbb{N}^+$, with $n|m!$. From the definition one can easily deduce that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ is the canonical prime factorization of n, then $S(n) = \max\{S(p_i^{\alpha_i})\}$, where the maximum is taken over the i's from 1 to k. This observation illustrates the importance of being able to calculate the Smarandache function for prime powers. This paper will be considering that process. We will give an upper and lower bound for $S(p^\alpha)$ in Theorem 1.4. A recursive procedure of calculating $S(p^\alpha)$ is then given in Proposition 1.8. Before preceeding we offer these trivial observations:

Observation 1. *If p is prime, then $S(p) = p$.*

Observation 2. *If p is prime, then $S(p^k) \leq kp$.*

Observation 3. *p divides $S(p^k)$*

Observation 4. *If p is prime and $k < p$, then $S(p^k) = kp$.*

To see that observation 4 holds, one need only consider the sequence

$$2, 3, 4 \dots, p-1, p, p+1, \dots, 2p, 2p+1, \dots, 3p, \dots, kp$$

and count the elements which have a factor of p.

Define $T_p(n) = \sum_{k=1}^{\infty} [\frac{n}{p^k}]$, where $[\cdot]$ represents the greatest integer function. The function T_p counts the number of powers of p in $n!$. To relate $T_p(n)$ and $S(n)$ note that $S(p^\alpha)$ is the smallest n such that $T_p(n) \geq \alpha$. In other words $S(p^\alpha)$ is characterized by

$$(*) \qquad T_p(S(p^\alpha)) \geq \alpha \quad \text{and} \quad T_p(S(p^\alpha) - 1) \leq \alpha - 1.$$

Lemma 1.0. *For $n \geq 1$, $T_p(n) < \frac{n}{p-1}$*

Proof. $T_p(n) = \sum_{k=1}^{\infty} [\frac{n}{p^k}] < \sum_{k=1}^{\infty} \frac{n}{p^k} = \frac{n}{p-1}$ □

Corollary 1.1. $(p-1)\alpha < S(p^\alpha) \leq p\alpha$

Recall this basic fact about the p-adic representation of a number n. Given $n, p \in \mathbb{Z}$ and $p \geq 2$, $n \geq 0$, we can uniquely represent $n = \sum_{j=0}^{\infty} a_j(n)p^j$, where each $a_j \in \{0, 1, 2, \ldots, p-1\}$.

Lemma 1.2. $T_p(n) = \frac{1}{p-1}(n - \sum_{j=0}^{\infty} a_j(n))$

Proof.

$$\begin{aligned}
T_p(n) &= \sum_{k=1}^{\infty} [\frac{n}{p^k}] = \sum_{k=1}^{\infty} \Big[\frac{\sum_{j=0}^{\infty} a_j(n)p^j}{p^k}\Big] \\
&= \sum_{k=1}^{\infty} \frac{\sum_{j=k}^{\infty} a_j(n)p^j}{p^k} = \sum_{k=1}^{\infty}\sum_{j=k}^{\infty} a_j(n)p^{j-k} \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{j} a_j(n)p^{j-k} = \sum_{j=1}^{\infty} a_j(n)\sum_{k=1}^{j} p^{j-k} \\
&= \sum_{k=1}^{\infty} a_k(n)\sum_{j=1}^{k} p^{k-j} = \frac{1}{p-1}\sum_{k=1}^{\infty} a_k(n)(p^k - 1) \\
&= \frac{1}{p-1}\sum_{k=1}^{\infty}(a_k(n)p^k - a_k(n)) \\
&= \frac{1}{p-1}(n - \sum_{k=0}^{\infty} a_k(n)) \quad \square
\end{aligned}$$

Lemma 1.3. *If $n \geq 1$ then*

$$1 \leq \sum_{j=0}^{\infty} a_j(n) \leq (p-1)[[\log_p(n)] + 1].$$

Proof. For each a_j we have $a_j \leq p-1$. Note that in the p-adic expansion of n, $a_j(n) = 0$ for all $j > [\log_p(n)]$. Thus we have $1 \leq \sum_{j=0}^{\infty} a_j(n) \leq (p-1)([\log_p(n)] + 1)$.

Now using the characterization $*$ and Lemma 1.2, we get the following

$$S(p^\alpha) - \sum_{j=0}^{\infty} a_j(S(p^\alpha)) \geq (p-1)\alpha \quad \text{and}$$

$$(**) \qquad S(p^\alpha) - 1 - \sum_{j=0}^{\infty} a_j(S(p^\alpha) - 1) \leq (\alpha - 1)(p-1).$$

Applying Lemma 1.3 to the first inequality for $S(p^\alpha)$, yields a lower bound of

$$S(p^\alpha) \geq (p-1)\alpha + 1.$$

This lower bound cannot be improved since we obtain equality when $\alpha = p+1$, in fact we achieve equality whenever $\alpha = p^t + p^{t-1} + \cdots + p + 1$ for $t \geq 1$. Now $S(p^\alpha)$ is clearly integer valued, so one may choose to write the lower bound as $S(p^\alpha) > (p-1)\alpha$.

From the latter inequality (**), we get the following.

$$\begin{aligned}
S(p^\alpha) &\leq (p-1)(\alpha-1) + 1 + \sum_{j=0}^{\infty} a_j(S(p^\alpha) - 1) \\
&\leq (p-1)(\alpha-1) + 1 + (p-1)([\log_p(S(p^\alpha) - 1)] + 1) \\
&= (p-1)(\alpha-1) + 1 + (p-1)[\log_p(S(p^\alpha) - 1)] + (p-1) \\
&= \alpha(p-1) + (p-1)[\log_p(S(p^\alpha) - 1)] + 1 \\
&\leq \alpha(p-1) + (p-1)[\log_p(p\alpha - 1)] + 1 \\
&\leq \alpha(p-1) + (p-1)[\log_p(p\alpha)] + 1 \\
&= \alpha(p-1) + (p-1)[\log_p(\alpha) + 1] + 1 \\
&= \alpha(p-1) + (p-1)[\log_p(\alpha)] + (p-1) + 1 \\
&= (p-1)[\alpha + 1 + \log_p(\alpha)] + 1
\end{aligned}$$

Theorem 1.4. *For any prime p and any integer α, we have*

$$(p-1)\alpha + 1 \leq S(p^\alpha) \leq (p-1)[\alpha + 1 + \log_p(\alpha)] + 1.$$

We now consider the sharpness of this upper bound. Note that when $\alpha = p^k - k$ the upper bound yields the value $(p-1)p^k + 1$. As it turns out $S(p^{p^k - k})$ is one less than this yield.

Lemma 1.5. *$S(p^{p^k-k}) = (p-1)p^k$, for $k \geq 1$.*

Proof. Consider

$$\begin{aligned} T_p(p^{k+1}-p^k) &= \sum_{l=1}^{\infty}\left[\frac{p^{k+1}-p^k}{p^l}\right] \\ &= (p^k - p^{k-1}) + (p^{k-1} - p^{k-2}) + \cdots + (p^2 - p) + (p-1) = p^k - 1 \end{aligned}$$

and

$$\begin{aligned} T_p(p^{k+1}-p^k-1) &= \sum_{l=1}^{\infty}\left[\frac{p^{k+1}-p^k-1}{p^l}\right] \\ &= \left[p^k - p^{k-1} - \frac{1}{p}\right] + \left[p^{k-1} - p^{k-2} - \frac{1}{p^2}\right] + \cdots + \left[1 - \frac{1}{p} - \frac{1}{p^{k+1}}\right] \\ &= (p^k - p^{k-1} - 1) + (p^{k-1} - p^{k-2} - 1) + \cdots + (p - 1 - 1) + 0 \\ &= p^k - (k+1). \end{aligned}$$

Since $T_p(p^{k+1}-p^k-1) < p^k - k \leq T_p(p^{k+1}-p^k)$, we have $S(p^{p^k-k}) = (p-1)p^k$. □

Thus we have produced infinitely many values that are within one of the upper bound. If we recall Observation 3, the upper bound should be congruent to 0 mod p. So one could subtract the remainder of the upper bound when dividing by p from the upper bound and make it sharp. We shall omit that task in this paper.

We now turn our attention to answering the question when is $S(p^\alpha) = p^\beta$. Consider the following calculations, verification is left for the reader.

$$T_p(p^{\beta+1}) = p^\beta + p^{\beta-1} + \cdots + p + 1$$

$$T_p(p^{\beta+1} - 1) = p^\beta + p^{\beta-1} + \cdots + p - \beta$$

$$T_p(p^\beta) = p^{\beta-1} + p^{\beta-2} + \cdots + p + 1$$

$$T_p(p^\beta - 1) = p^{\beta-1} + p^{\beta-2} + \cdots + p + 1 - \beta$$

Thus we have $S(p^\alpha) = p^{\beta+1}$ if $p^\beta + p^{\beta-1} + \cdots + p + 1 - \beta \leq \alpha \leq p^\beta + p^{\beta-1} + \cdots + p + 1$. If $p^{\beta-1} + p^{\beta-2} + \cdots + p + 1 \leq \alpha < p^\beta + p^{\beta-1} + \cdots + p + 1 - \beta$, then we have $p^\beta \leq S(p^\alpha) < p^{\beta+1}$.

We now offer a recursive procedure for calculating $S(p^\alpha)$. The following is a technical lemma that will be used in proving the recursion formula.

Lemma 1.6. *Suppose we have $p^\beta \le r < p^{\beta+1}$, for some $\beta \ge 0$, then $T_p(r) = T_p(p^\beta) + T_p(r - p^\beta)$.*

Proof.

$$\begin{aligned} T_p(r) = \sum_{k=1}^{\infty} [\frac{r}{p^k}] &= \sum_{k=1}^{\beta} [\frac{p^\beta + (r - p^\beta)}{p^k}] \\ &= \sum_{k=1}^{\beta} (\frac{p^\beta}{p^k}) + \sum_{k=1}^{\beta} [\frac{r - p^\beta}{p^k}] \\ &= T_p(p^\beta) + T_p(r - p^\beta) \quad \square \end{aligned}$$

Lemma 1.7. *If $p^{\beta-1} + p^{\beta-2} + \cdots + p + 1 \le \alpha < p^\beta + p^{\beta-1} + \cdots + p + 1$, then $S(p^\alpha) = p^\beta + S(p^{\alpha - (p^{\beta-1} + p^{\beta-2} + \cdots + p + 1)})$.*

Proof. Case 1: Assume that $p^{\beta-1} + p^{\beta-2} + \cdots + p + 1 \le \alpha < p^\beta + p^{\beta-1} + \cdots + p + 1 - \beta$.

$$\begin{aligned} S(p^\alpha) &= \min\{r | T_p(r) \ge \alpha\} \\ &= \min\{r | T_p(r) \ge \alpha \quad \text{and} \quad p^\beta \le r < p^{\beta+1}\} \\ &= \min\{r | T_p(p^\beta) + T_p(r - p^\beta) \ge \alpha \quad \text{and} \quad p^\beta \le r < p^{\beta+1}\} \\ &= p^\beta + \min\{r - p^\beta | T_p(r - p^\beta) \ge \alpha - T_p(p^\beta) \quad \text{and} \quad 0 \le r - p^\beta < p^{\beta+1} - p^\beta\} \\ &= p^\beta + \min\{r | T_p(r) \ge \alpha - T_p(p^\beta) \quad \text{and} \quad 0 \le r < p^{\beta+1} - p^\beta = p^\beta(p-1)\} \\ &= p^\beta + S(p^{\alpha - T_p(p^\alpha)}) \\ &= p^\beta + S(p^{\alpha - (p^{\beta-1} + p^{\beta-2} + \cdots + p + 1)}) \end{aligned}$$

Case 2: Assume that $p^\beta + p^{\beta-1} + \cdots + p + 1 - \beta \le \alpha < p^\beta + p^{\beta-1} + \cdots + p + 1$. From the prior calculations of $T_p(p^{\beta+1})$ and $T_p(p^{\beta+1} - 1)$ we have the $S(p^\alpha) = p^{\beta+1}$ for any α in this range. Now consider the right hand side of the equation, $p^\beta + S(p^{\alpha - (p^{\beta-1} + p^{\beta-2} + \cdots + p + 1)})$. We can restate this expression as $p^\beta + S(p^t)$, where $p^\beta - \beta \le t < p^\beta$. From the proof of Lemma 1.4 we see that $T_p(p^{\beta+1} - p^\beta) = p^\beta - 1$ and $T_p(p^{\beta+1} - p^\beta - 1) = p^\beta - \beta - 1$, thus it must be that $S(p^t) = p^{\beta+1} - p^\beta$. Therefore the right hand side is $p^{\beta+1}$. $\square$

Clearly this lemma can be repeated as long as $\alpha - (p^{\beta-1} + \ldots 1) \geq p^{\beta-1} + \ldots 1$, so we can strengthen Lemma 1.6.

Proposition 1.8. *If* $d = p^{\beta-1} + p^{\beta-2} + \cdots + p + 1 \leq \alpha < p^{\beta} + p^{\beta-1} + \cdots + p + 1$, *write* $\alpha = qd + r$ *with* $0 \leq r < d$, *then* $S(p^{\alpha}) = qp^{\beta} + S(p^{r})$.

Now $p^{\beta} + p^{\beta-1} + \cdots + p + 1 = p^{\beta}(1 + \frac{1}{p} + \cdots + \frac{1}{p^{\beta}}) \leq \frac{p^{\beta+1}}{p-1}$. Therefore we get $\log_p \alpha < \log_p(p^{\beta} + \cdots + 1) = \beta + 1 - \log_p(p-1) < \beta + 1$, and similarly $\beta - 1 < \beta - \log_p(p-1) < \log_p(\alpha) < \beta + 1$, or $\log_p \alpha - 1 < \beta < \log_p \alpha + 1$. Hence the exact value of $S(p^{\alpha})$ can be obtained by applying the proposition on the order of $\log_p \alpha$ times.

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Critical Trigger Mechanism – a Modeling Paradigm for Cognitive Science Application in the Design of Artificial Learning Systems

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ABSTRACT

In our present *short paper* we introduce a rather promising modeling paradigm for the design of artificial learning systems, incorporating *critical trigger mechanism (CTM)*. We contend that at various stages of the learning process, such trigger mechanism may be activated when certain 'critical' points in the learning curve are attained. Such points are marked by *fuzzification* of the learner's decision set. At all other 'non-critical' points where the decision set is crisp, this trigger mechanism lies dormant. We proceed to show that identification and subsequent incorporation of such trigger mechanisms will be of substantial help in modeling learning systems that closely emulate cognitive learning pattern of the human mind. This is not a complete work in any sense but just an indication of what is to come - a mere map of the long and challenging road ahead.

Key Words

Artificial Learning Systems, Fuzzy Logic, Cognitive Science, Directed and Non-directed Interventions

Introduction

The conditioned-reflex experiments of the Russian physiologist Ivan Pavlov and the American psychologist Edward Thorndike were central to the development of behaviorist model of learning. However, modern cognitive science favors a logical-computational model of learning over the rather mechanistic stimulus-response model of traditional behaviorism. But there need not exist as big a chasm between the approach of traditional behaviorism and that of modern cognitive psychology as is often made out to be. Gagne

and Briggs (1974) have already attempted to combine behaviorist principles of learning with a cognitive theory of learning named Information-Processing. They believe that the design of intervention must be undertaken with suitable attention to the conditions under which learning occurs.

Information-Processing theory regards human learning as being analogous to a computer and its ability to store memory. Significant efforts have already been made to design artificial systems that emulate human learning and memory. In this regard, the ***Memory Extender (ME)*** personal filing system design is an illustrative example that immediately springs to mind. As humans we process information initially with our senses. This information is either processed into short-term memory or is lost. If this information is continually re-used it is processed into long-term memory. However, for this information processing there has to be some initial directed interventions ***(hard programming)*** followed by subsequent non-directed interventions ***(soft programming)***. At times, these two forms of intervention may become mutually inconsistent. It is especially to deal with such situations that we suggest the incorporation of *critical trigger mechanism (CTM)*, in order to make the system decide upon a definite course of action.

The Proposed Modeling Paradigm

Let us consider a case where an artificial learning system is being trained to emulate investor behavior. The fundamental operational rule which the system needs to learn is a simple IF statement – ***"Buy IF price is rising AND Sell IF price is falling"***. But simply learning this fundamental rule may not enable the system to realistically emulate the actual behavior of a human investor. The fundamental rule is nevertheless important – it is the initial hard programming bit consisting of a directed intervention. This is the easy part. But for a realistic simulation, the system must also learn to do some internal cognitive processing in accordance with one or more subsequent non-directed interventions – the soft programming bit.

If we are trying to design a system to emulate an individual investor's fund allocation behavior then we have to prima facie consider the subtle cognitive factors underlying such behavior over and above those dictated by hard economic reasoning. The boundary between the preference sets of an individual investor, for funds allocation between a risk-free asset and the risky market portfolio, tends to be rather fuzzy as the investor continually evaluates and shifts his or her position; unless it is a passive *buy-and-hold* kind of portfolio.

Thus, if the universe of discourse is $\mathbf{U} = \{\mathbf{C, N, A}\}$ where C, N and A are three risk classes **"conservative"**, **"neutral"** and **"aggressive"** respectively, then the fuzzy subset of U given by $\mathbf{P} = \{x_1/\mathbf{C}, x_2/\mathbf{N}, x_3/\mathbf{A}\}$ is the true preference set for our purposes. Here we have $\mathbf{0} \leq (x_1, x_2, x_3) \leq \mathbf{1}$, all the symbols having their usual meanings. Although theoretically any of the $\mathbf{P}(x_i)$ values could be equal to unity, in reality it is far more likely that $\mathbf{P}(x_i) < \mathbf{1}$ **for** $\mathbf{i = 1, 2, 3}$ i.e. the fuzzy subset P is most likely to be ***subnormal.*** Also, similarly, in most real-life cases it is expected that $\mathbf{P}(x_i) > \mathbf{0}$ **for** $\mathbf{i = 1, 2, 3}$ i.e. all the elements of P will be included in its support: $\mathbf{supp\ (P) = \{C, N, A,\} = U}$.

The critical point of analysis is definitely the individual investor's preference ordering i.e. whether an investor is primarily conservative or primarily aggressive. It is understandable that a primarily conservative investor could behave aggressively at times and vice versa but in general, their behavior will be in line with their classification. So the classification often depends on the height of the fuzzy subset P: **height (P) = $\mathbf{Max_x P(x)}$**. So one would think that the risk-neutral class becomes largely superfluous, as investors in general will tend to get classified as either primarily conservative or primarily aggressive. However, as already said, in reality, the element N will also generally have a non-zero degree of membership in the fuzzy subset and hence cannot be dropped.

The fuzziness surrounding investor classification stems from the fuzziness in the preference relations regarding the allocation of funds between the risk-free and the risky assets in the optimal portfolio. It may be mathematically described as follows:

Let M be the set of allocation options open to the investor. Then, the fuzzy preference relation is a fuzzy subset of the **M x M** space identifiable by the following membership function:

$$\mu R(m_i, m_j) = 1;\ m_i \text{ is definitely preferred to } m_j$$
$$c \in (0.5, 1);\ m_i \text{ is somewhat preferred to } m_j$$
$$0.5;\ \text{point of perfect neutrality}$$
$$d \in (0, 0.5);\ m_j \text{ is somewhat preferred to } m_i;\ \text{and}$$
$$0;\ m_j \text{ is definitely preferred to } m_i$$

The fuzzy preference relation is assumed to meet the necessary conditions of reciprocity and transitivity. Then a CTM would be a built-in function in conjunction with the above membership function, such that, when activated, it would instantaneously convert the fuzzy preference relation into a crisp preference relation.

As long as a subsequent soft programming is consistent with the initial hard programming, the decision set will be crisp: the universe of discourse and the crisp decision subsets being of the following form:

$$D = \{d_1, d_2 \ldots d_i \ldots d_n\};$$
$$d = \{d_1, d_2 \ldots d_i \ldots d_k, (d_i \in D) \cap (d_i \notin d^c)\},$$
$$d^c = \{d_{k+1}, d_{k+2} \ldots d_{k+l} \ldots d_m, (d_{k+l} \in D) \cap (d_{k+l} \notin d)\},\ \text{such that } d \cap d^c = \phi$$

However, at a point of conflict between the initial hard programming and a subsequent soft programming, the decision set will be fuzzified with an unchanged universe of discourse but fuzzy decision subsets of the following form:

$$D = \{d_1, d_2 \ldots d_i \ldots d_n\};$$
$$d = \{p_1/d_1, p_2/d_2 \ldots p_i/d_i \ldots p_n/d_n, (d_i \in D), (0 \le p_i \le 1)\},$$
$$d^c = \{q_1/d_1, q_2/d_2 \ldots q_i/d_i \ldots q_n/d_n, (d_i \in D), (0 \le q_i \le 1)\},$$
$$\text{such that } d \cap d^c \neq \phi$$

Therefore, any function having the potential to be a CTM must be having the following fundamental characteristics:

- **It should be activated if and only if the decision set is fuzzified at any stage in the learning process**
- **It should, when activated, convert a fuzzy decision set into a crisp decision set**
- **It should mark a critical point on the system learning curve by either advancing or setting back the learning process**

Suppose a novice investor goes on putting more and more of his or her funds in a particular asset just because it has been steadily outperforming the market index over the recent past. Then, suddenly one fine day the bubble bursts and our investor is left in the red with the greater part of his or her equity wiped out. How far will that investor be inclined to invest in a similar asset in the distant future when such type of assets are doing great once again? Economic reasoning (hard programming) will encourage the investor to go with the trend and once again start putting his or her funds on that asset. But the investor's cognitive process (soft programming) may not be in tune with the directed intervention of market economics. This would ***fuzzify*** the decision set for the investor. This is where a potential CTM could be activated which ultimately decides which way the investor would go by ***de-fuzzifying*** the decision set.

In case of our investor, if the CTM activation actually hinders learning then he or she will be inclined to leave that offending asset alone no matter how lucrative an investment opportunity seems. If on the other hand the CTM activation actually facilitates learning then the investor will go for that asset once again but adopt a more circumspect approach – having positively ***learned*** from his or her previous misadventure. However, in either case, the CTM has the effect of ***de-fuzzifying*** the investor's decision set.

The extent of potential impact of the CTM could also be effectively modeled as a fuzzy function characterized by the universe of discourse $\{C_S, C_M, C_W\}$ corresponding to **"strong"**, **"moderate"** or **"weak"** impact respectively, with the governing fuzzy subset $\{\theta_1/C_S, \theta_2/C_M, \theta_3/C_W\}$; $(0 \leq \theta_1, \theta_2, \theta_3 \leq 1)$. An artificial learning system would have an advantage in this regard as such a system could incorporate the different possible forms (at varying strengths of impact) of the CTM and perform a ***what-if*** analysis to see exactly how different the individual outcomes are in each case.

The Road Ahead

What we have here is some kind of a hypothesis regarding modeling of artificial learning systems that emulate the human learning process. As our next step we plan to identify a potential CTM in human learning behavior specifically in relation to investing. One prime candidate we feel could be the ***post-investment cognitive dissonance factor*** due to inconsistency in perceived and true worth of an investment, which can and often do critically affect an investor's learning behavior. Subsequently, we propose to incorporate this mechanism in a ***hybrid neuro-fuzzy system*** and emulate investor behavior under different market settings. If results are satisfactory then the approach could be extended to models covering other facets of human learning behavior. Finally we would need an ***effective integration strategy*** to bring the various models together in a unified whole. Once this integration is achieved over a fairly large area of human learning, we shall have

moved one significant step forward in creating the ultimate of all artificial learning systems – a working model of the human mind.

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THERE ARE INFINITELY MANY SMARANDACHE DERIVATIONS, INTEGRATIONS AND LUCKY NUMBERS

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Abstract

A number is said to be a *Smarandache Lucky Number* (see [3, 1, 2]) if an incorrect calculation leads to a correct result. In general, a *Smarandache Lucky Method or Algorithm* is said to be any incorrect method or algorithm, which leads to a correct result. In this note we find an infinite sequence of *distinct* lucky fractions. We also define a lucky product differentiation and a lucky product integration. For a given function f, we find *all* other functions g, which renders the product lucky for differentiation/integration.

Keywords. Smarandache Lucky Numbers, Fractions, Lucky Derivatives, Lucky Integrals

1 Introduction

A number is said to be a *Smarandache Lucky Number* (see [2]) if an incorrect calculation leads to a correct result. For example, in the fraction 64/16 if the 6's are incorrectly cancelled the result $4/1 = 4$ is correct. (We exclude trivial examples of the form 400/200 where non-aligned zeros are cancelled.)

In general: *The Smarandache Lucky Method/Algorithm/Operation/etc.* is said to be any incorrect method or algorithm or operation, which leads to a correct result. The wrong calculation is funny, and somehow similarly to the students' common mistakes, or to produce

confusions or paradoxes. In [1] (see also [2], [3]), the authors ask the questions: *Is the set of all fractions, where an incorrect calculation leads to a correct result, finite or infinite? Can someone give an example of a Smarandache Lucky Derivation, or Integration, or Solution to a Differential Equation?*

In this note we give an infinite class of examples of each type. In fact, given a real-valued function f, we find all examples for which an incorect differentiation/integration, in a product with f, leads to a correct answer.

2 Main Results

Let f, g be real-valued functions. Define the incorect differentiation as follows:
$\frac{d_O\,(f(x)\cdot g(x))}{dx} = \frac{df(x)}{dx}\cdot\frac{dg(x)}{dx}$.
We prove

Theorem 1. *Let $f : \mathbf{R} \to \mathbf{R}$. The functions $g : \mathbf{R} \to \mathbf{R}$, satisfying $\frac{d_O(f(x)\cdot g(x))}{dx} = \frac{df(x)\cdot g(x)}{dx}$ are given by*

$$g(x) = c\cdot e^{\int \frac{f'(x)}{f'(x)-f(x)}dx},$$

where c is a real constant.

Proof. Since $\frac{d_O(f(x)\cdot g(x))}{dx} = f'(x)g'(x)$, we need to find all functions such that

$$f'(x)g'(x) = f'(x)g(x) + f(x)g'(x),$$

by the product rule for differentiation. Thus, we need

$$g'(x)(f'(x)-f(x)) = g(x)f'(x) \iff \frac{g'(x)}{g(x)} = \frac{f'(x)}{f'(x)-f(x)},$$

from which we derive

$$g(x) = c\cdot e^{\int \frac{f'(x)}{f'(x)-f(x)}dx}.$$

□

Examples.

1. Take $f(x) = x$, then $g(x) = c \cdot e^{\int \frac{1}{1-x} dx} = c \cdot e^{\ln(1/(1-x))} = c \cdot \frac{1}{1-x}$.
2. Take $f(x) = x^2$, then

$$g(x) = c \cdot e^{\int \frac{2x}{2x - x^2} dx} = c \cdot e^{\ln \frac{x}{x-2} + \ln \frac{1}{x^2 - 2x}} = \frac{c}{(x-2)^2}.$$

From the previous theorem we derive an equivalent result on lucky integration. The incorect integration is defined by: *the integral of a product is the product of integrals.*

Theorem 2. *Given a real-valued function f, the functions g such that the integral of the product of f and g is the product of the integral of f and integral of g are given by*

$$g(x) = \frac{c f(x)}{f(x) - \int f(x) dx} \cdot e^{\int \frac{f(x)}{f(x) - \int f(x) dx} dx}.$$

Proof. Similar to the proof of Theorem 1. □

Obviously, the previous theorem is an example of a lucky differential equation, as well.

3 There Are an Infinity Number of ... Lucky Numbers

To avoid triviality, we exclude among the lucky numbers, those which are constructed by padding at the end the same number of zeros in the denominator and numerator of a fixed fraction (e.g., $\frac{3000}{11000}$). We also exclude 1's, that is $\frac{ab\cdots x}{ab\cdots x}$.

The fact that there are an infinity of lucky fractions is not a difficult question (even if they are not constructed by padding zeros or they come from 1). Our next result proves that

Theorem 3. *Let the fraction* $\frac{99\cdots96}{24\cdots99}$ *(same number of digits). By cancelling as many* 9*'s as we wish (and from any place, for that matter), we still get* 4.

Proof. Let $n+1$ be the number of digits in the numerator (or denominator) of the given fraction. We write it as

$$\begin{aligned}&\frac{9\cdot10^n+9\cdot10^{n-1}+\cdots+9\cdot10+6}{24\cdot10^{n-1}+9\cdot10^{n-2}+\cdots+9}\\&=\frac{3\cdot\frac{10^n-4}{9}}{8\cdot10^{n-2}+3\cdot\frac{10^{n-2}-1}{10-1}}=\frac{10^n-4}{24\cdot10^{n-2}+10^{n-2}-1}\\&=\frac{10^n-4}{\frac{1}{4}10^n-1}=4.\end{aligned}$$

We see that by cancelling any number of digits of 9, we get a fraction of the same form. □

In the same manner we can show (we omit the proof)

Theorem 4. *Define the fractions* $\frac{33\cdots32}{8\cdots33}$ *(the numerator has one digit more that the denominator), respectively,* $\frac{6\cdots64}{16\cdots6}$ *(same number of digits),* $\frac{9\cdots95}{19\cdots9}$ *(same number of digits),* $\frac{6\cdots65}{26\cdots6}$ *(same number of digits),* $\frac{9\cdots98}{49\cdots9}$ *(same number of digits),* $\frac{77\cdots75}{217\cdots7'}$, $\frac{13\cdots34}{3\cdots34}$ *(same number of* 3*'s). By cancelling as many* $3's$*, respectively,* $6's, 9's, 6's, 9's, 7's, 3's$*, as we wish, we get the same number, namely* 4*, respectively,* $5, \frac{5}{2}, 2, \frac{25}{7}, 4$.

Other examples of lucky numbers are given by taking the above fractions and inserting zeros appropriately. We give

Theorem 5. *The following fractions are also lucky numbers*

$$\frac{b0\cdots0xy}{a0\cdots0wz}$$

(same number of zeros), where $1\le a,b,w,x,y,z\le9$ *are integers,* $\frac{xy}{wz}$ *are the fractions from the previous theorem equal to* $\{2,5/2,4,5\}$ *and* $\frac{b}{a}$ *is equal to that same reduced fraction. When* $\frac{xy}{wz}=\frac{25}{7}$*, then* a,b *are not digits, rather they are integers such that* $\frac{b}{a}=\frac{25}{7}$.

You might think that these are the only lucky numbers. That is not so. Our last theorem will present an infinite number of distinct *lucky* numbers.

Theorem 6. *Take* any *reduced fraction* $\frac{b}{a}$. *Then, the following sequence of fractions is a sequence of lucky numbers* $\frac{b0\cdots0b}{a0\cdots0a}$. *Assuming the denominator (numerator) has k more digits than the numerator (denominator), then the numerator (denominator) has k more zeros in it. Since* $\frac{b}{a}$ *was arbitrary, we have an infinite number of lucky fractions.*

Example. Let $\frac{b}{a} = \frac{11}{7}$. Then we build the following sequence of lucky numbers

$$\frac{11}{7}, \frac{11011}{7007}, \frac{110011}{70007}, etc.$$

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PROGRAM FOR FINDING OUT NUMBER OF SMARANDACHE DISTINCT RECIPROCAL PARTITION OF UNITY OF A GIVEN LENGTH

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ABSTRACT: Smarandache Distinct Reciprocal partition of unity for a given length '**n**' is defined as the number of ways in which unity can be expressed as the sum of the reciprocals of '**n**' distinct numbers. In this note a program in 'C' is given.

// This is a program for finding number of distinct reciprocal partitions of unity of a given length written by K Suresh, Software expert, IKOS , NOIDA , INDIA.

```
#include<stdio.h>
#include<math.h>
unsigned long TOTAL;

FILE* f;
long double array[100];
unsigned long count = 0;

void try(long double prod, long double sum, unsigned long pos)
{
        if( pos == TOTAL - 1 )
        {
                // last element..
                long double diff = prod - sum;
                if( diff == 0 ) return;

                array[pos] = floorl(prod / diff);
                if( array[pos] > array[pos-1] && array[pos] * diff == prod )
                {
                 fprintf(f, "(%ld) %ld", ++count,(unsigned long)array[0]);
                 int i;
                 for(i = 1; i < TOTAL; i++) fprintf(f,", %ld", (unsigned long)array[i]);
                 fprintf(f, "\n");
                 fflush(f);
```

```
            }
            return;
        }
        long double i;
        if( pos == 0)
         i = 1;
        else
         i = array[pos-1];

        while(1) {
        i++;
        long double new_prod = prod * pow(i, TOTAL-pos);
        long double new_sum = (TOTAL-pos) * (new_prod / i);
        unsigned long j;
        for(j = 0; j < pos; j++) new_sum += new_prod / array[j];
        if( new_sum < new_prod )
              break;

        new_prod = prod * i;
        array[pos] = i;
        new_sum = prod + sum * i;
        if( new_sum >= new_prod ) continue;

        try(new_prod, new_sum, pos+1);
       }
       return;
}

main()
{
      printf("Enter no of elements ?");
      scanf("%ld", &TOTAL);
      char fname[256];
      sprintf(fname, "rec%ld.out", TOTAL);
      f = fopen(fname, "w");
      fprintf(f, "No of elements = %ld.\n", TOTAL);

     try(1, 0, 0);
     fflush(f);
     fclose(f);
     printf("Total %ld solutions found.\n", count);
```

```
    return 0;
}
```

Based on the above program the following table is formed.

Length	Number of Distinct Reciprocal Partitions
1	1
2	0
3	1
4	6
6	2320
7	245765

Reference:

[1] " Amarnath Murthy" , ' Smarandache Reciprocal Partition of Unity sets and sequences' , SNJ, Vol. 11, No. 1-2-3 , 2000.

On a problem concerning the Smarandache friendly prime pairs

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Abstract

In this paper a question posed in [1] and concerning the Smarandache friendly prime pairs is analysed.

Introduction

In [1] the Smarandache friendly prime pairs are defined as those prime pairs (p,q) such that:

$$\sum_{x=p}^{q} x = p \cdot q \qquad (1)$$

where x denote the primes between p and q. In other words the Smarandache friendly prime pairs are the pairs (p,q) such that the sum of the primes between p and q is equal to the product of p and q.
As example let's consider the pair (2,5). In this case $2+3+5=2\cdot 5$ and then 2 and 5 are friendly primes. The other three pairs given in the mentioned paper are: (3,13), (5,31) and (7,53). Then the following open questions have been posed:

Are there infinitely many friendly prime pairs?

Is there for every prime p a prime q such that (p,q) is a Smarandache friendly prime pair?

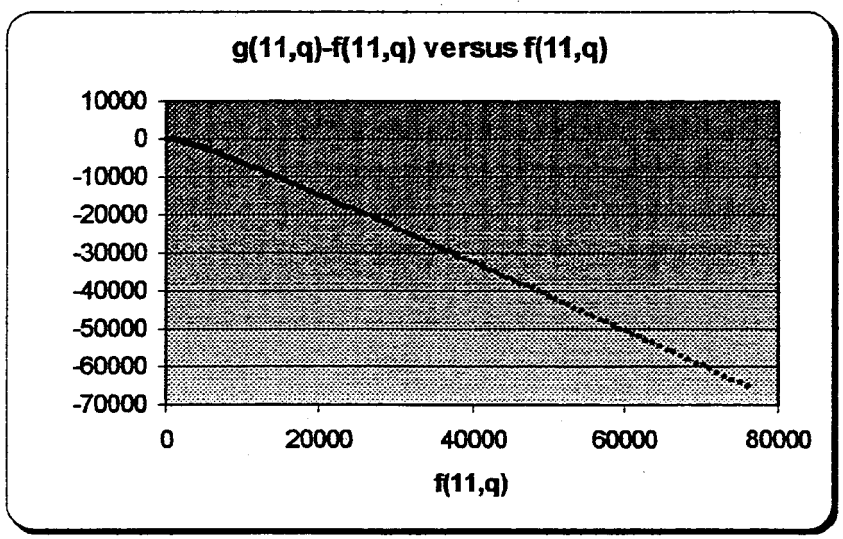
In this paper we analyse the last question and a shortcut to explore the first conjecture is reported.

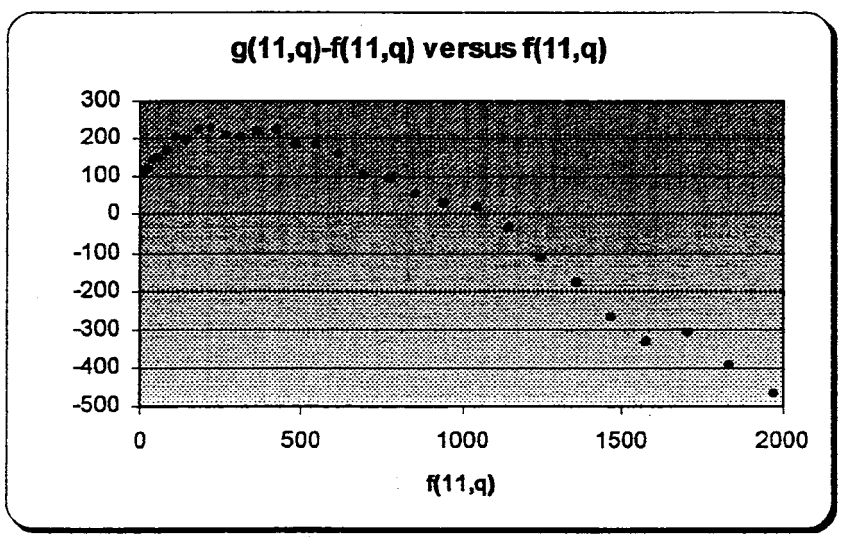
Results

First of all let's analyse the case p=11. Let's indicate:

$$f(11,q)=\sum_{x=11}^{q} x \quad and \quad g(11,q)=11\cdot q$$

where x denotes always the primes between 11 and q.
A computer program with Ubasic software package has been written to calculate the difference between g(11,q) and f(11,q) for the 164 primes q subsequent to 11. Here below the trend of that difference.





As we can see the difference starts to increase, arrives to a maximum and then starts to decrease and once pass the x axis decrease in average linearly. The same thing is true for all the other primes p.
So for every prime p the search of its friend q can be performed up to:

$$g(p,q) - f(p,q) \leq -M$$

where M is a positive constant.

For the first 1000 primes M has been choosen equal to 10^5.
No further friendly prime pair besides those reported in [1] has been found. According to those experimental results we are enough confident to pose the following conjecture:

Not all the primes have a friend, that is there are prime p such that there isn't a prime q such that the (1) is true .

Moreover a furter check of friendly prime pairs for all primes larger than 1000 and smaller than 10000 has been performed choosing M=1000000.
No further friendly prime pair has been found. Those results seem to point out that the number of friendly prime pairs is finite.

Question:

Are (2,5), (3,13), (5,31) and (7,53) the only Smarandache friendly prime pairs?

References.

[1] A. Murthy, *Smarandache friendly numbers and a few more sequences*, Smarandache Notions Journal, Vol. 12 N. 1-2-3 Spring 2001

SMARANDACHE SEQUENCE OF HAPPY NUMBERS

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Abstract:

In this article, we present the results of investigation of Smarandache Concatenate Sequence formed from the sequence of Happy Numbers and report some primes and other results found from the sequence

Key words:

Happy numbers, Consecutive happy numbers, H-sequence, Smarandache H-sequence, Reversed Smarandache H-sequence, Prime, Happy prime, Reversed Smarandache Happy Prime, Smarandache Happy Prime

1. Introduction:

If you iterate the process of summing the squares of the decimal digits of a number and if the process terminates in 1, then the original number is called a Happy number [1].

For example:

7 -> 49 -> 97 -> 130 -> 10 -> 1, so the number 7 is a happy number.

Let us denote the sequence of Happy numbers as H-sequence. The sequence of Happy numbers [3], say H = { 1, 7, 10, 13, 19, 23, 28, 31, 32, 44, 49, 68, 70, 79, 82, 86, 91, 97 ,100 .....}.

2. Smarandache Sequence:

Let S_1 , S_2 , S_3 , . . . , S_n , . . . be an infinite integer sequence (termed as S- sequence), then the Smarandache sequence [4] or Smarandache Concatenated sequence [2] or Smarandache S-sequence is given by

$S_1,\ S_1S_2,\ S_1S_2S_3 \ \ldots \ S_1S_2S_3 \ldots S_n \ldots$

Also Smarandache Back Concatenated sequence or Reversed Smarandache S-sequence is

$S_1,\ S_2S_1,\ S_3S_2S_1 \ \ldots \ S_n \ldots \ S_3S_2S_1 \ldots$

3. Smarandache H-Sequence:

Smarandache sequence of Happy numbers or Smarandache H-sequence is the sequence formed from concatenation of numbers in H-sequence (Note that H-sequence is the sequence of Happy numbers). So, Smarandache H-sequence is

1, 17, 1710, 171013, 17101319, 1710131923, 171013192328, ...

Let us denote the n^{th} term of the Smarandache H-sequence by SH(n). So,

SH(1)=1

SH(2)=17

SH(3)=1710

SH(4)=171013 and so on.

3.1 Observations on Smarandache H-sequence:

We have investigated Smarandache H-sequence for the following two problems.

i. How many terms of Smarandache H-sequence are primes?

ii. How many terms of Smarandache H-sequence belongs to the initial H-sequence?

In search of answer to these problems, we find that

a. There are only 3 primes in the first 1000 terms of Smarandache H-sequence. These are SH(2) = 17, SH(5) = 17101319 and SH(43), which is 108 digit prime. It may be noted that SH(1000) consists of 3837 digits.

Open Problem:

Can you find more primes in Smarandache H-sequence and are there infinitely many such primes?

b. There are 1429 Happy numbers in first 10000 terms of Smarandache H-sequence and hence belongs to the initial H-sequence. The first few Happy numbers in the Smarandache H-sequence are SH(1), SH(11), SH(14), SH(30), SH(31), SH(35), SH(48), SH(52), SH(62), SH(67), SH(69), SH(71), SH(76), ..., etc.

It may be noted that SH(10000) consists of 48396 digits.

Based on the investigations we state the following:

Conjecture:

About one-seventh of numbers in the Smarandache H-sequence belong to the initial H-sequence.

In this connection, it is interesting to note that about one-seventh of all numbers are happy numbers [1].

3.2 Consecutive SH Numbers:

It is known that smallest pair of consecutive happy numbers is 31, 32. The smallest triple is 1880, 1881, 1882. The smallest example of four and 5 consecutive happy numbers are 7839, 7840, 7841, 7842 and 44488, 44489, 44490, 44491, 44492 respectively. Example of 7 consecutive happy numbers is also known [3]. The question arises as to how many consecutive terms of Smarandache H-sequence are happy numbers.

Let us define consecutive SH numbers as the consecutive terms of Smarandache H-sequence which are happy numbers. During investigation of first 10000 terms of Smarandache H-sequence, we found the following smallest values of consecutive SH numbers:

Smallest pair: SH(30) , SH(31)

Smallest triple: SH(76), SH(77), SH(78)

Smallest example of four and five consecutive SH numbers are SH(153), SH(154), SH(155), SH(156) and SH(3821), SH(3822), SH(3823), SH(3824), SH(3825) respectively.

Open Problem:

Can you find the examples of six and seven consecutive SH numbers?

How many consecutive SH numbers can you have?

4.0 Reversed Smarandache H-Sequence:

It is defined as the sequence formed from the concatenation of happy numbers (H-sequence) written backward i.e. in reverse order. So, Reversed Smarandache H-sequence is

1, 71, 1071, 131071, 19131071, 2319131071, 282319131071, ...

Let us denote the n^{th} term of the Reversed Smarandache H-sequence by RSH(n). So,

RSH(1)=1

RSH(2)=71

RSH(3)=1071

RSH(4)=131071 and so on.

4.1 Observations on Reversed Smarandache H-sequence:

Since the digits in each term of Reversed Smarandache H-sequence are same as in Smarandache H-sequence, hence the observations regarding problem (ii) including conjecture mentioned in para 3.1 above remains valid in the present case also. So, only observations regarding problem (i) mentioned in para 3.1 above are given below:

As against only 3 primes in Smarandache H-sequence, we found 8 primes in first 1000 terms of Reversed Smarandache H-sequence. These primes are:

RSH(2) = 71

RSH(4) = 131071

RSH(5) = 19131071

RSH(6) = 2319131071

RSH(10) = 443231282319131071

Other three primes are RSH(31), RSH(255) and RSH(368) which consists of 72, 857 and 1309 digits respectively.

Smarandache Curios:

It is interesting to note that there are three consecutive terms in Reversed Smarandache H-sequence, which are primes, namely RSH(4), RSH(5) and RSH(6), which is rare in any Smarandache sequence.

We also note that RSH(31) is prime as well as happy number , so, this can be termed as Reversed Smarandache Happy Prime. No other happy prime is noted in Reversed Smarandache H-sequence and Smarandache H-sequence.

Open Problem:

Can you find more primes in Reversed Smarandache H-sequence and are there infinitely many such primes?

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On a Smarandache problem concerning the prime gaps

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Abstract
In this paper, a problem posed in [1] by Smarandache concerning the prime gaps is analysed.

Let's p_n be the n-th prime number and d_n the following ratio:

$$d_n = \frac{p_{n+1} - p_n}{2} \quad \text{where} \quad n \geq 1$$

If we indicate with $g_n = p_{n+1} - p_n$ the gap between two consecutive primes, the previous equation becomes:

$$d_n = \frac{g_n}{2}$$

In [1], Smarandache posed the following questions:

1. Does the sequence d_n contain infinite primes?
2. Analyse the distribution of d_n

First of all let's observe that d_n is a rational number only for n=1, being $p_1 = 2$, $p_2 = 3$. For $n > 1$, instead, the ratio is always a natural number since the gap of prime numbers g_n is an even number ≥ 2 [2].
Moreover let's observe that the gap g_n can be as large as we want. In fact let's n be any integer greater than one and let's consider the following sequence of consecutive integers:

$$n!+2,\ n!+3,\ n!+4, \ldots\ldots, n!+n$$

Notice that 2 divides the first, 3 divides the second, ..., n divides the n-1st, showing all of these numbers are composite. So if p is the largest prime smaller than $n!+2$ we have $g_n > n$. This proves our assertion.
Now let's check the first terms of sequence d_n:

n	1	2	3	4	5	6	7	8	9	10
d	0.5	1	1	2	1	2	1	2	3	1
p_n	2	3	5	7	11	13	17	19	23	29

Here p_n is the smallest prime relative to the gap d_n. As we can see, for the first 10 terms of sequence d_n we have 4 primes regardless if those are repeated or not. On the contrary, if we consider only how many distinct primes we have then this number is 2. So, the Smarandache question can be split in two sub-questions:

1. How many times the sequence d_n takes a prime value?
2. How many distinct primes the sequence d_n contains?

Proving both the questions is a very difficult task. Anyway, we can try to understand the behaviour of sequence d_n by using a computer search and then get a heuristic argument on the number of primes within it.
Thanks to an Ubasic code, the counting functions $p_1(N)$ and $p_2(N)$ have been calculated for N up to 10^9.
$p_1(N)$ denotes how many times d_n takes a prime value for $n \le N$ while $p_2(N)$ denotes the number of distinct primes in d_n, always for $n \le N$. In table 1, the results of the computer search can be found. In the third column, the number of distinct primes are reported whereas in the second one the number of all primes regardless of the repetitions are shown.

N	# primes	# distinct primes
10	0	0
100	14	2
1000	107	4
10000	695	7
100000	4927	11
1000000	37484	14
10000000	241286	19
100000000	2413153	24
1000000000	66593597	33

Table 1. Number of primes in d_n for different N values

Let's analyse the data of column 2. It is very easy to verify that those data grow linearly with N, that is:

$$p_1(N) \approx c(N) \cdot N \qquad (1)$$

An estimation of c(N) can be obtained using the following asymptotic relationship given in [3]:

$$h_N(d_n) \approx \frac{c_2 \cdot N}{\ln^2(N)} \cdot \prod_{p \mid 2d_n,\ p>2} \frac{p-1}{p-2} \cdot e^{-\frac{2 \cdot d_n}{\ln(N)}}$$

where $h_N(d_n)/N$ is the frequency of d_n for $n \le N$ and p any prime number.
The constant c_2 is the twin prime constant defined in the following way:

$$c_2 \equiv 2 \cdot \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) = 1.320032 \ .....$$

By definition of $p_1(N)$ function we have:

$$p_1(N)=\sum_{d_n=2}^{d_{max}} \frac{c_2\cdot N}{\ln^2(N)}\cdot \prod_{p|2d_n,\ p>2}\frac{p-1}{p-2}\cdot e^{-\frac{2\cdot d_n}{\ln(N)}} \quad (2)$$

where the above summation is extended on all prime values of d_n up to d_{max}. But the largest gap d_{max} for a given N can be approximated by [2],[3]:

$$d_{max}\approx\frac{1}{2}\cdot\ln^2(N)$$

and then (2) can be rewritten as:

$$p_1(N)\approx \frac{2\cdot N}{\ln^2(N)}\sum_{d_n=2}^{\frac{1}{2}\ln^2(N)} e^{-\frac{2\cdot d_n}{\ln(N)}} \quad (3)$$

where the function :

$$J(d_n)=\prod_{p|2d_n,\ p>2}\frac{p-1}{p-2}$$

has small values of order 1 and then has been replaced by its mean value $\frac{2}{c_2}$ [3].

Since, as N goes to infinity, the summation:

$$\sum_{d_n=2}^{\frac{1}{2}\ln^2(N)} e^{-\frac{2\cdot d_n}{\ln(N)}}$$

is the number of primes in the range 1 to $\frac{1}{2}\cdot\ln^2(N)$, we can write:

$$\sum_{d_n=2}^{\frac{1}{2}\ln^2(N)} e^{-\frac{2\cdot d_n}{\ln(N)}}\le\pi(\frac{1}{2}\cdot\ln^2(N))$$

where $\pi(N)$ is the counting function of prime numbers [2]. Using the Gauss approximation [2] for it, we have:

$$\sum_{d_n=2}^{\frac{1}{2}\ln^2(N)} e^{-\frac{2\cdot d_n}{\ln(N)}} \leq \frac{\frac{1}{2}\cdot\ln^2(N)}{\ln(\frac{1}{2}\cdot\ln^2(N))}$$

and then:

$$p_1(N) \approx c(N)\cdot N \leq \frac{N}{\ln\left(\frac{1}{2}\cdot\ln^2(N)\right)}$$

by using (1) and (3), that implies:

$$c(N) \leq \frac{1}{\ln\left(\frac{1}{2}\cdot\ln^2(N)\right)}$$

According to those experimental data the following conjecture can be posed:

Conjecture A: The sequence d_n takes infinite times a prime value.

Let's now analyse the data reported in table 1, column 3. By using the least square method, we can clearly see that the best fit is obtained using a logarithmic function like:

$$p_2(N) \approx c(N)\cdot\ln(N) \qquad (4)$$

where c(N) can be estimated using the following approximation:

$$p_2(N) \approx \pi(0.5\cdot\ln^2(N))$$

being $p_2(N)$ the number of primes in the range 1 to d_{max}.
Therefore:

$$\frac{\ln^2(N)}{2\cdot\ln(0.5\cdot\ln^2(N))} \approx c(N)\cdot\ln(N)$$

$$\Rightarrow \quad c(N) \approx \frac{\ln(N)}{2\cdot\ln(0.5\cdot\ln^2(N))}$$

In table 2, the comparison of (4) with calculated values $p_2(N)$ shown in table 1 (column 3) is reported. Notice the good agreement between $p_2(N)$ and its estimation as N increase.
According to those data, also $p_2(N)$ like $p_1(N)$ goes to the infinity as N increase, although $p_2(N)$ more slowly then $p_1(N)$. Then this second conjecture can be posed:

Conjecture B: The sequence d_n contains an infinite number of distinct primes

$$\frac{\ln^2(N)}{2\cdot\ln(0.5\cdot\ln^2(N))} \approx c(N)\cdot\ln(N)$$

$$\Rightarrow \quad c(N) \approx \frac{\ln(N)}{2\cdot\ln(0.5\cdot\ln^2(N))}$$

In table 2, the comparison of (4) with calculated values $p_2(N)$ shown in table 1 (column 3) is reported. Notice the good agreement between $p_2(N)$ and its estimation as N increase.
According to those data, also $p_2(N)$ like $p_1(N)$ goes to the infinity as N increase, although $p_2(N)$ more slowly then $p_1(N)$. Then this second conjecture can be posed:

Conjecture B: The sequence d_n contains an infinite number of distinct primes

N	$P_2(n)$	$c(N)\cdot\ln(N)$	ratio
10	0	2.719152	0
100	2	4.490828	0.445352
1000	4	7.521271	0.531825
10000	7	11.31824	0.618471
100000	11	15.80281	0.696079
1000000	14	20.93572	0.668713
10000000	19	26.69067	0.711859
100000000	24	33.04778	0.726221
1000000000	33	39.9911	0.825184

Table2. Comparison of $p_2(N)$ with the approximated formula $c(N)\cdot\ln(N)$. In the third column the ratio $p_2(N)/c(N)\cdot\ln(N)$

Let's analyse now the distribution of d_n, as always requested by Smarandache.
Thanks to a Ubasic code the frequency of prime gaps up to N=3601806621 have been calculated.
The plot of those frequencies versus d_n for $n>1$ is reported in Fig1. It shows a clear jigsaw pattern superimposed onto an exponential decay. The jigsaw pattern is due to a double population that is clearly visible in the two plots of fig 2.
The frequency of d_n for n being a multiple of 3 (or equivalently for n multiple of 6 for g_n) is always larger than adjacentes differences.

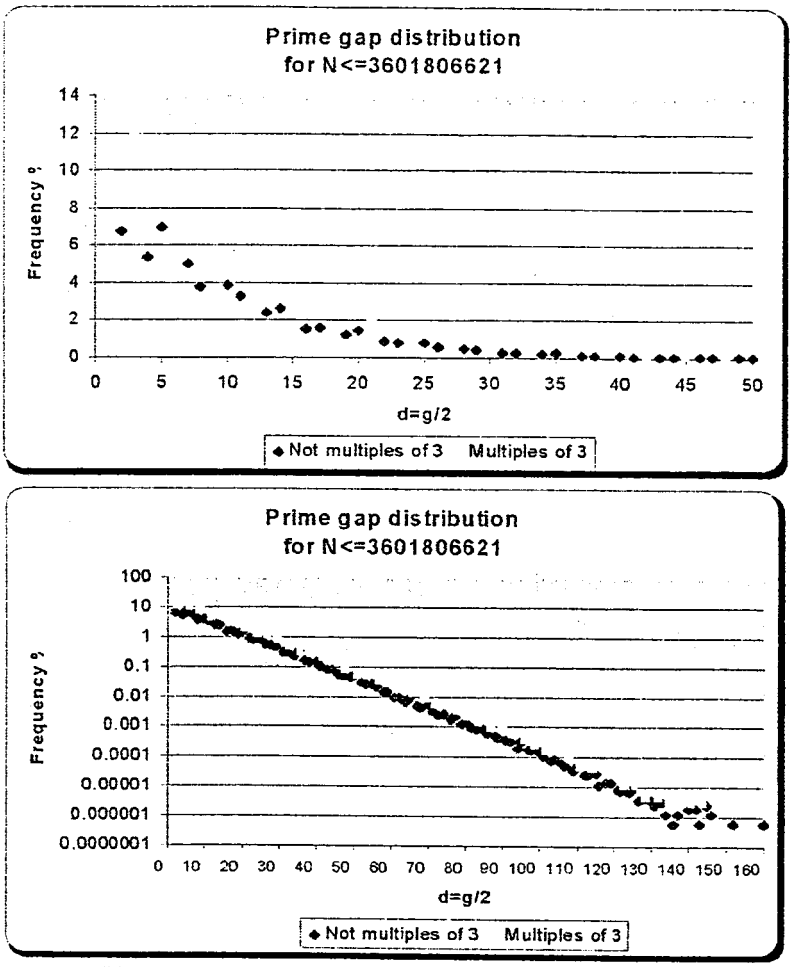

Fig 2. Prime gap distribution. The second plot uses a logarithmic scale for the Y-axis.

According to the conjecture 1 reported in [3] and already mentioned above , the number of pairs p_n, $p_{n+1} < N$ with $d_n = \frac{p_{n+1} - p_n}{2}$ is given by:

$$h_N(d_n) \approx \frac{c_2 \cdot N}{\ln^2(N)} \cdot \prod_{p|2d_n,\ p>2} \frac{p-1}{p-2} \cdot e^{-\frac{2 \cdot d_n}{\ln(N)}}$$

Let's $f(p) = \frac{p-1}{p-2}$ where p is any prime number greater than 2. As it can be seen in fig 3. this function approaches 1 quickly, with the maximum value at p=3.

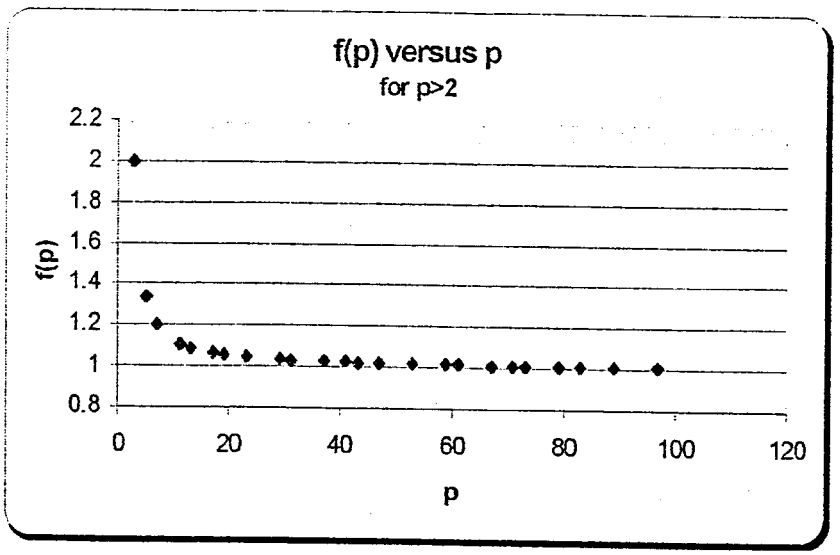

Fig.3: Plot of function f(p) versus p

Being f(p) maximum for p=3 means that $h_N(d_n)$ has a relative maximum every time $2d_n$ has 3 as prime factor, that is when $2d_n$ is a multiple of 3.
This explains the double population seen in the Fig 2 and then the jigsaw pattern of the fig 1.
In fig. 4, the distribution of d_n obtained by computer search and the one estimated with the use of $h_N(d_n)$ formula is reported . Notice the very good agreement between them.

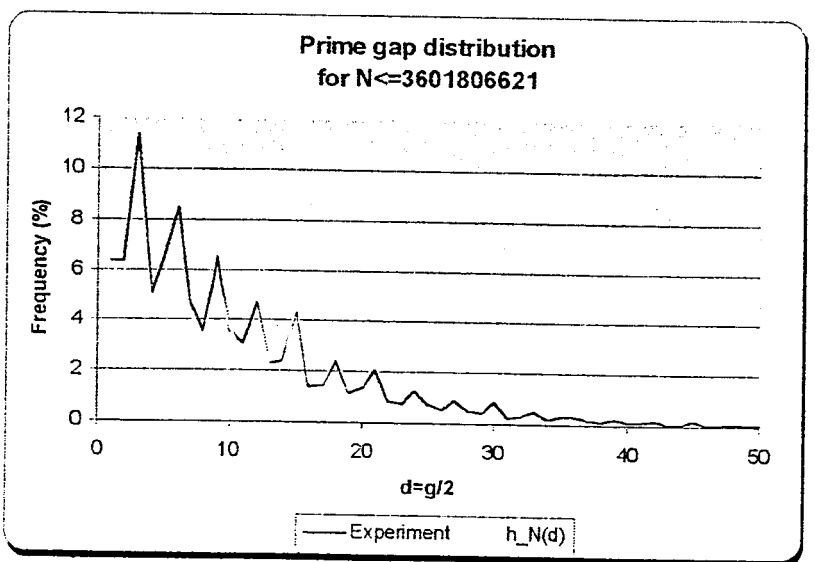

Fig 4: Prime gap distribution comparison. The good agreement between the experimental and the estimated data has to be noticed.

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JAVA CONCURENT PROGRAM FOR THE SAMARANDACHE FUNCTION

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Abstract: The aim of this article is to propose a Java concurrent program for the Smarandache function based on the equation $S(p_1^{k_1}\cdot\ldots\cdot p_s^{k_s})=\max\{S(p_1^{k_1}),\ldots,S(p_s^{k_s})\}$. Some results concerning the theoretical complexity of this program are proposed. Finally, the experimental results of the sequential and Java programs are given in order to demonstrate the efficiency of the concurrent implementation.

1. INTRODUCTION

In this section the results used in this article are presented briefly. These concern the Smarandache and the main methods of its computation. The Smarandache function [Smarandache, 1980] is $S: N^* \to N$ defined by

$$S(n)=\min\{k\in N | k! \vdots n\}\ (\forall n\in N^*). \tag{1}$$

The main properties of this function are presented in the following

$$(\forall a,b\in N^*)\,(a,b)=1 \Rightarrow S(a\cdot b)=\max\{S(a),S(b)\} \tag{2}$$

that gives us

$$S(p_1^{k_1}\cdot\ldots\cdot p_s^{k_s})=\max\{S(p_1^{k_1}),\ldots,S(p_s^{k_s})\}. \tag{3}$$

An important inequality satisfied by the function S is

$$(\forall a\in N^*)\ S(a)\le a\text{, the equality occurring } \textit{iff } a \text{ is prime.} \tag{4}$$

When the number a is not prime this inequality can be improved by $(\forall a\in N^*: \text{a not prime})\ S(a)\le \frac{a}{2}$.

During the last few years, several implementation of The Smarandache function have been proposed. Ibstedt [1997, 1999] developed an algorithm based on Equation (3). The

implementation in U Basic provided a efficient and useful program for computing the values of S for large numbers. Based on it Ibstedt [1997, 1999] studied several conjectures on the Smarandache function. No study of the theoretical complexity has provided for this algorithm so far.

The second attempt to develop a program for the Smarandache function was made by Tabirca [1997]. Tabirca started from Equation (1) and considered the sequence $x_k = k! \bmod n$. The first term equal to 0 provides the value $S(n)$. Unfortunately, the C++ implementation of this algorithm has been proved not to be useful because it cannot be applied for large value of n. Furthermore, this is not an efficient computation because the value $S(n)$ is computed in $O(S(n))$. A study of the average complexity [Tabirca, 1997a, 1998], [Luca, 1999] gave that the average complexity of this algorithm is $O\left(\frac{n}{\log n}\right)$.

2. AN EFFICIENT ALGORITHM FOR THE SMARANDACHE FUNCTION

In this section we develop an efficient version of the algorithm proposed by Ibsedt. A theoretical study of this algorithm is also presented. Equation (3) reduces the computation of $S(n)$ to the computation of the values $S(p_i^{k_i}), i = 1,...,s$. The equation [Smarandache, 1980] that gives the value $S(p^k)$ is given by

$$k = \sum_{i=1}^{l} d_i \cdot \frac{p^i - 1}{p-1} \Rightarrow S(p^k) = \sum_{i=1}^{l} d_i \cdot p^i . \tag{5}$$

This means that if $(d_l, d_{l-1},...,d_1)$ is the representation of k in the generalized base $1, \frac{p^2-1}{p-1},...,\frac{p^l-1}{p-1}$, then $(d_l, d_{l-1},...,d_1)$ is the representation of $S(p^k)$ k in the generalized base $p, p^2,...,p^l$. Denote $b1[i] = \frac{p^i - 1}{p-1}$ and $b2[i] = p^i$ the general terms of these two bases. We remark that the terms of the above generalized bases satisfied:

$$b1[1] = 1, b1[i+1] = 1 + p \cdot b1[i] \tag{6}$$

$$b2[1] = p, b2[i+1] = p \cdot b2[i]. \tag{7}$$

```
public static long Value (final long p, final long k) {
        long l, j, value=0;
        long b1[] = new long [1000]; long b2[] = new long [1000];
        b1[0]=1;b2[0]=p;
        for(int l=0;b1[l]<=k;l++){b1[l+1]=1+p*b1[l]; b2[l+1]=p*b2[l];}
        for(l--,j=l;j>=0;j--){d=p/b1[j];p=p%b1[j];value+=d*b2[j]; }
        return value;
}
```

Figure 1. Java function for $S(p^k)$.

Equation (5) provides an algorithm that is presented in Figure 1. At the first stage this algorithm finds the largest l such that $b1[l] \le k < b1[l+1]$ and computes the generalized bases $b1$ and $b2$. At the second stage the algorithm determines the representation of k in the base $b1$ and the value of this representation in the base $b2$.

Theorem 1. *The complexity of the computation* $S(p^k)$ *is* $O(\log_p p \cdot k)$.

Proof. Let us remark that the operation number of the function Value is $5 \cdot l$, where l is the largest value such that $b1[l] \le k < b1[l+1]$. This gives the following equivalences

$$\frac{p^l - 1}{p-1} \le k < \frac{p^{l+1} - 1}{p-1} \Leftrightarrow p^l - 1 \le k \cdot (p-1) < p^{l+1} - 1 \Leftrightarrow$$

$$\Leftrightarrow p^l \le k \cdot (p-1) + 1 < p^{l+1} \Leftrightarrow l \le \log_p [k \cdot (p-1) + 1] < l + 1 \Leftrightarrow$$

$$\Leftrightarrow l = \lfloor \log_p (k \cdot (p-1) + 1) \rfloor.$$

Therefore, the number of operations is $5 \cdot \lfloor \log_p (k \cdot (p-1) + 1) \rfloor = O(\log_p (k \cdot p))$. ♦

The computation of $S(n)$ is obtained in two steps. Firstly, the prime number decomposition $n = p_1^{k_1} \cdot \ldots \cdot p_s^{k_s}$ is determined and all the values $S(p_i^{k_i}), i = 1,\ldots,s$ are found by using a calling of the function Value. Secondly, the maximum computation is used to find $\max\{S(p_1^{k_1}),\ldots,S(p_s^{k_s})\}$. A complete description of this algorithm is presented in Figure 2.

```
public static long S (final long n) {
        long d, valueMax=0, s=-1;
        if (n= =1) return 0;
        long p[] = new long [1000]; long k[] = new long [1000]; long value[] = new long [1000];
        for(d=2;d<n;d++) if (n % d == 0){
                s++;p[s]=d;for(k[s]=0;n%d==0;k[s]++,n/=d);
                value[s]=Value(p[s],k[s]);
                }
        for(j=0;j<=s;j++) if (valueMax<value[j])valueMax=value[j];
        return valueMax;
}
```

Figure 2. Java function for $S(n)$.

Theorem 2. *The complexity of the function S is* $O(\frac{n}{\log n})$.

Proof. In order to find the prime number decomposition, all the prime numbers less than n should be checked. Thus, at most $\pi(n) = O(\frac{n}{\log n})$ checking operations are performed [Bach & Shallit, 1996] to find the prime divisors $p_1,...,p_s$ of n. The exponents $k_1,...,k_s$ of these prime numbers are found by $k_1 + ... + k_s$ divisions. An upper bound for this sum is obtained as follows

$$n = p_1^{k_1} \cdot ... \cdot p_s^{k_s} \Rightarrow \log n = \log p_1^{k_1} \cdot ... \cdot p_s^{k_s} = \log p_1^{k_1} + ... + \log p_s^{k_s} =$$

$$= k_1 \cdot \log p_1 + ... + k_s \cdot \log p_s \geq k_1 + ... + k_s,$$

because each logarithm is greater than 1. Thus, we have $k_1 + ... + k_s \leq \log n = O(\log n)$.

The computation of all the values $S(p_i^{k_i}), i = 1,...,s$ gives a complexity equal to $\sum_{i=1}^{s} \log_{p_i} p_i \cdot k_i$. An upper bound for this sum is provided by the following inequality

$\log_{p_i} p_i \cdot k_i \le k_i$ that is true because of $p_i \cdot k_i \le p_i^{k_i}$. Taking the sum we find $\sum_{i=1}^{s} \log_{p_i} p_i \cdot k_i \le \sum_{i=1}^{s} k_i = O(\log n)$, therefore the complexity of this computation is $O(\log n)$.

Finally, observe that the maximum $\max\{S(p_1^{k_1}),...,S(p_s^{k_s})\}$ is determined in $s \le \log n$ operations.

In conclusion, the complexity of the Smarandache function computation is $O(\frac{n}{\log n})$. ♦

	n=10000	n=20000	n=30000	n=40000	n=50000	n=60000	n=70000	n=80000
A 1	2804	10075	21411	36803	56271	79304	105922	136567
A 2	2925	10755	23284	39967	61188	86555	115837	149666

Table 1. Running times for the efficient and Tabirca's algorithms.

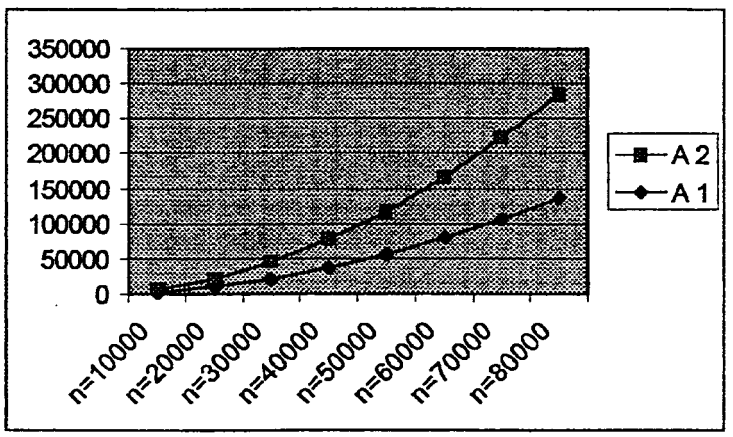

Figure 3. Graphics of the Running Times.

Several remarks can be made after this theorem. Firstly, we have found that finding the prime divisors of n represents the most expensive operation and this gives the complexity of the function computation. Secondly, we have obtained an algorithm with the complexity $O\left(\frac{n}{\log n}\right)$. Therefore, this is better than the algorithm proposed by Tabirca [1988] that has the average complexity $O\left(\frac{n}{\log n}\right)$. Table 1 shows that this algorithm also offers better running times than the

algorithm proposed in [Tabirca, 1997]. These two algorithms were implemented in Java and executed on PENTIUM II machine. The times [milliseconds] of the computation for all the values $S(i)$, i=1,...,n were found, where n=10000,....,80000. Row A 1 gives the times for this efficient algorithm and row A 2 gives the times for the algorithm proposed in [Tabirca, 1999]. Another important remark drawn from Table 1 is that the difference between the times of each column does not increase faster [see Figure 3]. This is happen because the complexity of the algorithm proposed by Tabirca [1997] is $O\left(\frac{n}{\log n}\right)$.

3. JAVA CONCURRENT ALGORITHM FOR THE SMARANDACHE FUNCTION

In this section we present a Java concurrent program for the computation described in Section 2. Firstly, remark that many operations of this algorithm can be performed in parallel. Consider that we know all the prime numbers less than n. Usually, this can be done by using special libraries. Let $p_1,...,p_s$ be these numbers. Therefore, we can concurrently execute the computation of the exponent of p_i and the computation of the value $S(p_i^{k_i})$.

A Java program may contain sections of code that are executed simultaneously. An independent section of code is known as a thread or lightweight process [Smith, 1999]. The implementation presented here is based on equation (3): $S(p_1^{k_1}\cdot...\cdot p_s^{k_s})=\max\{S(p_1^{k_1}),...,S(p_s^{k_s})\}$. Each $S(p_i^{k_i})$ is calculated concurrently in a thread. On single processor systems, the use of threads simulates the concurrent execution of some piece of sequential code. The worst case execution time can be taken as the longest execution time for a single thread. On a multi processor system, given enough processors, each thread should ideally be allocated to a processor. If there are not enough processors available, threads will be allocated to processors in groups. Unlike pure concurrent processes, threads are used to simulate concurrency within a single program. Most current everyday programs use threads to handle different tasks. When we click a save icon on a word processing document typically a thread is created to handle the actual saving action. This allows the user to continue working on the document while another process (thread in this case) is writing the file to disk.

For the concurrent algorithm consider the Java function for $S(n)$ in Figure 2. Typical areas that can be executed concurrently can be found in many loops, where successive iterations of the loop

do not depend on results of previous iterations. In Figure 4, we adapt the for loop (Figure 2) to execute the Value function (Figure 1), responsible for calculating $S(p^k)$, concurrently by creating and executing a ValueThread object. When all the required threads have begun execution, the value of max will not be known until they have completed. To detect this, a simple counter mechanism is employed. As threads are created the counter is incremented and as threads complete their tasks the counter is decremented. All threads are completed when this counter reaches 0.

```
public long S(long n)
{
        if (n==1) return (long);

        Prime decom = new Prime(n);
        noPrimes=decom.noPrime();
        if (noPrimes == 0)
                value = null;
        value = new long[noPrimes];

        for (int k=0;k<noPrimes;k++)
        {
                started++;
                new ValueThread(decom.getPrime(k), decom.getPow(k), this, k);
        }

        while (started > 0)
        {
                try
                {
                        Thread.yield();
                }catch (Exception e)
                {
                }
        }
        return max;
}
```

Figure 4. Modified Java function for $S(n)$, used to concurrently execute the Value function

As each thread completes its task it executes a callback method, addValue (Figure 5). This method is declared as synchronized to prevent multiple threads calling the addValue method at the same time. Should this be allowed to occur, an incorrect value of the number of threads executing would be created. Execution of this method causes the value array declared in method

S (Figure 3) to be filled. This value array will only be completely filled after the last thread makes a call to the addValue method. At this point, the value of max can be determined.

```
public synchronized void addValue(int k, long val)
{
        value[k] = val;
        max = value[0];
        started--;
        if (started == 0)
                for (int i=1; i<=k; i++)
                        if (value[i] > max)
                                max = value[i];
  }
```

Figure 5. The addValue method called by a Thread when its task is completed.

This algorithm illustrates how concurrency can be employed to improve execution time. It is also possible to parallelise the algorithm at a higher level, by executing the function responsible for calculating each $S(n)$ in an independent thread also. Tests of this mechanism however show that it is more efficient to only parallelise the execution of $S(p^t)$.

The concurrent Java program has been run on a SGI Origin 2000 parallel machine with 16 processors. The execution was done with 1, 2, 4 processors only and the execution times are shown in Table 1. The first line of Table 1 shows the running times for Algorithm A1 on this machine. The next three lines present the running times for the concurrent Java program when p=1, p=3 and p=4 processors are used.

	n=20000	n=30000	n=40000	n=50000	n=60000	n=70000	n=80000
A 1	9832	19703	31237	49774	68414	96242	115679
CA (p=1)	9721	19474	30195	49412	68072	95727	115161
CA (p=2)	5786	11238	22872	31928	42825	60326	75659
CA (p=4)	3863	7881	14017	19150	30731	42508	53817

Table 2. Running Times for the Concurrent Program.

4. CONCLUSSIONS

Several remarks can be drawn after this study. Firstly, Equation (3) represents the source of any efficient implementation of the Smarandache function. In Section 2 we have proposed a sequential algorithm with the complexity $O\left(\frac{n}{\log n}\right)$. We have also proved both theoretically and practically that this algorithm is better that the algorithm developed in [Tabirca, 1997].

Secondly, we have developed a Java concurrent program in order to decrease the computation time. Based on the thread technique we have performed concurrently the computation of the values $S(p_i^{k_i})$. This concurrent implementation has proved to be better than the sequential one. Even running with one single processor the times of the concurrent Java program were found better than the times of the sequential program.

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Appendix A

The full code for the concurrent implementation presented in Section 3.

```
// Smarandache.java

import java.io.*;
import java.util.*;

public class Smarandache
{
        public Smarandache()
        {
                long n=0, i, j;
                long val;
                BufferedReader br = new BufferedReader(new InputStreamReader(System.in));
                try
                {
                        System.out.print ("n = ");
                        n = Integer.parseInt(br.readLine());
                }catch (IOException e)
                {
                        System.out.println ("IOException : "+e.getMessage());
                        System.exit(1);
                }

                Smar sm = new Smar();

                Date begin = new Date();
                for (i=1; i<= n; i++)
                {
                        val = sm.S(i);
                }
                Date end = new Date();
                System.out.println ("Time good is "+ (end.getTime() - begin.getTime()));
        }

        public static void main (String args[])
        {
                new Smarandache();
        }
}
```

```
// Smar.java

public class Smar
{
        private long value[];
        private long max = Long.MIN_VALUE;
        private int noPrimes=0;
        private int started = 0;

        public Smar()
        {
        }

        public long S(long n)
        {
                if (n==1)
                        return (long) 0;

                Prime decom = new Prime(n);
                noPrimes=decom.noPrime();
                if (noPrimes == 0)
                        value = null;
                value = new long[noPrimes];

                for (int k=0;k<noPrimes;k++)
                {
                        started++;
                        new ValueThread(decom.getPrime(k), decom.getPow(k), this, k);
                }

                while (started > 0)
                {
                        try
                        {
                                Thread.yield();
                        }catch (Exception e)
                        {
                        }
                }
                return max;
        }
```

```
        public synchronized void addValue(int k, long val)
        {
                value[k] = val;
                started--;
                if (started == 0)
                {
                        max = value[0];
                        for (int i=1; i<=k; i++)
                                if (value[i] > max)
                                        max = value[i];
                }
        }
}
```

```
//Prime.java

public class Prime
{
        private int s;
        private long p[]=new long [1000];
        private int ord[]=new int [1000];

        public Prime()
        {
                s=0;
        }

        public Prime(long n)
        {
                long d;
                for(d=2,s=0;d<=n;d++)
                if(n%d == 0)
                {
                        p[s]=d;
                        for(ord[s]=0;;ord[s]++,n=n/d){if(n%d!=0)break;};
                        s++;
                }
        }

        public int noPrime()
        {
                return s;
        }

        public long getPrime(int i)
        {
                return p[i];
        }
```

```
	public int getPow(int i)
	{
		return ord[i];
	}
}
```

```
// ValueThread.java

public class ValueThread
{
	private long p=0, a=0;
	private Smar owner;
	private int index = 0;

	public ValueThread (long p, long a, Smar owner, int index)
	{
		this.p = p;
		this.a = a;
		this.owner = owner;
		this.index = index;
		run();
	}

	public long pseuPow(long p, long a)
	{
		if (a == 1)
			return (long) 1;
		return 1+p*pseuPow(p,a-1);
	}

	public long Pow(long p, long a)
	{
		if (a == 1)
			return (long) p;
		return p*Pow(p,a-1);
	}

	public void run()
	{
		long rest=a, val=0;
		int k, i;
		for(k=1;pseuPow(p,k)<=a;k++);k--;
		for(i=k;i>0;i--)
		{
			val += Pow(p,i)* (long)(rest / pseuPow(p,i));
			rest %= pseuPow(p,i);
		}
		owner.addValue(index, val);
	}
}
```

AN INTRODUCTION TO THE SMARANDACHE GEOMETRIES

by L. Kuciuk[1] and M. Antholy[2]

Abstract:
In this paper we make a presentation of these exciting geometries and present a model for a particular one.

Introduction:
An axiom is said *Smarandachely denied* if the axiom behaves in at least two different ways within the same space (i.e., validated and invalided, or only invalidated but in multiple distinct ways).

A *Smarandache Geometry* is a geometry which has at least one Smarandachely denied axiom (1969).

Notations:
Let's note any point, line, plane, space, triangle, etc. in a smarandacheian geometry by s-point, s-line,
s-plane, s-space, s-triangle respectively in order to distinguish them from other geometries.

Applications:
Why these hybrid geometries? Because in reality there does not exist isolated homogeneous spaces, but a mixture of them, interconnected, and each having a different structure.

In the Euclidean geometry, also called parabolic geometry, the fifth Euclidean postulate that there is only one parallel to a given line passing through an exterior point, is kept or validated.
In the Lobachevsky-Bolyai-Gauss geometry, called hyperbolic geometry, this fifth Euclidean postulate is invalidated in the following way: there are infinitely many lines parallels to a given line passing through an exterior point.
While in the Riemannian geometry, called elliptic geometry, the fifth Euclidean postulate is also invalidated as follows: there is no parallel to a given line passing through an exterior point.

Thus, as a particular case, Euclidean, Lobachevsky-Bolyai-Gauss, and Riemannian geometries may be united altogether, in the same space, by some Smarandache geometries. These last geometries can be partially Euclidean and partially Non-Euclidean. Howard Iseri [3] constructed a model for this particular Smarandache geometry, where the Euclidean fifth postulate is replaced by different statements within

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the same space, i.e. one parallel, no parallel, infinitely many parallels but all lines passing through the given point, all lines passing through the given point are parallel.

Let's consider Hilbert's 21 axioms of Euclidean geometry. If we Smarandachely deny one, two, three, and so on, up to 21 axioms respectively, then one gets:

$$_{21}C_1 + {}_{21}C_2 + {}_{21}C_3 + \ldots + {}_{21}C_{21} = 2^{21} - 1 = 2{,}097{,}151$$

Smarandache geometries, however the number is much higher because one axiom can be Smarandachely denied in multiple ways.
Similarly, if one Smarandachely denies the axioms of Projective Geometry, etc.

It seems that Smarandache Geometries are connected with the Theory of Relativity (because they include the Riemannian geometry in a subspace) and with the Parallel Universes (because they combine separate spaces into one space only) too.

A *Smarandache manifold* is an n-D manifold that supports a smarandacheian geometry.

Examples:
As a particular case one mentions *Howard's Models* [3] where a *Smarandache manifold* is a 2-D manifold formed by equilateral triangles such that around a vertex there are 5 (for elliptic), 6 (for Euclidean), and 7 (for hyperbolic) triangles, two by two having in common a side. Or, more general, an n-D manifold constructed from n-D submanifolds (which have in common two by two at most one m-D frontier, where $m<n$) that supports a Smarandache geometry.

A Mode for a particular Smarandache Geometry:
Let's consider an Euclidean plane (α) and three non-collinear given points A, B, and C in it. We define as s-points all usual Euclidean points and s-lines any Euclidean line that passes through one and only one of the points A, B, or C. Thus the geometry formed is smarandacheian because two axioms are Smarandachely denied:

a) The axiom that through a point exterior to a given line there is only one parallel passing through it is now replaced by two statements: one parallel, and no parallel.
Examples:
Let's take the Euclidean line AB (which is not an s-line according to the definition because passes through two among the three given points A, B, C), and an s-line noted (c) that passes through s-point C and is parallel in the Euclidean sense to AB:
- through any s-point not lying on AB there is one s-parallel to (c).
- through any other s-point lying on the Euclidean line AB, there is no s-parallel to (c).

b) And the axiom that through any two distinct points there exist one line passing through them is now replaced by: one s-line, and no s-line.
Examples:
Using the same notations:
- through any two distinct s-points not lying on Euclidean lines AB, BC, CA, there is one s-line passing through them;
- through any two distinct s-points lying on AB there is no s-line passing through them.

Miscellanea:

First International Conference on Smarandache Geometries will be held, between May 3-5, 2003, at the Griffith University, Queensland, Australia, organized by Dr. Jack Allen.
Conference's page is at: http://at.yorku.ca/cgi-bin/amca-calendar/public/display/conference_info/fabz54.
And it is announced at http://www.ams.org/mathcal/info/2003_may3-5_goldcoast.html as well.

There is a club too on "Smarandache Geometries" at http://clubs.yahoo.com/clubs/smarandachegeometries and everybody is welcome.

For more information see: http://www.gallup.unm.edu/~smarandache/geometries.htm or http://ca.geocities.com/mikeantholy/geometries.htm.

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SMARANDACHE SEMIRINGS AND SEMIFIELDS

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Abstract

In this paper we study the notions of Smarandache semirings and semifields and obtain some interesting results about them. We show that not every semiring is a Smarandache semiring. We similarly prove that not every semifield is a Smarandache semifield. We give several examples to make the concept lucid. Further, we propose an open problem about the existence of Smarandache semiring S of finite order.

Keywords: semiring, semifield, semi-algebra, distributive lattice, Smarandache semirings.

Definition [1] :

A non-empty set S is said to be a *semiring* if on S is defined two binary closed operations + and × such that (S, +) is an abelian semigroup with 0 and (S, ×) is a semigroup and multiplication distributes over addition from the left and from the right.

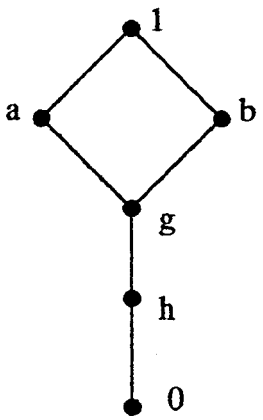
A semiring is a *strict semiring* if $x + y = 0$ implies $x = y = 0$. Semiring is *commutative* if (S, ×) is a commutative semigroup. A commutative semiring is a semifield if (S, ×) has a unit element and $x \times y = 0$ in S if and only if $x = y = 0$. For more properties of semirings please refer [1] , [3] , [4] and [5].

Definition 1:

The *Smarandache semiring* is defined [4] to be a semiring S such that a proper subset A of S is a semifield (with respect to the same induced operation). That is $\phi \neq A \subset S$.

Example 1: Let $M_{n\times n} = \{(a_{ij})/a_{ij} \in Z^+ \cup \{0\}\}$. Here, Z^+ denotes the set of positive integers. Clearly $M_{n\times n}$ is a semiring with the matrix addition and matrix multiplication. For consider $A = \{(a_{ij}) \mid a_{ij} = 0, i \neq j \text{ and } a_{ii} \in Z^+ \cup \{0\}\}$, that is all diagonal matrices with entries from $Z^+ \cup \{0\}$. Clearly, A is a semifield. Hence $M_{n\times n}$ is a Smarandache semiring.

Example 2: Let S be the lattice given by the following figure. Clearly S is a semiring under min-max operation. S is a Smarandache semiring for A = {1, b, g, h, 0} is a semifield.



Theorem 2:

Every distributive lattice with 0 and 1 is a Smarandache Semiring.

Proof: Any chain connecting 0 and 1 is a lattice which is a semifield for every chain lattice is a semiring which satisfies all the postulates of a semifield. Hence the claim.

Definition 3:

The *Smarandache sub-semiring* [4] is defined to be a Smarandache semiring B which is a proper subset of the Smarandache semiring S.

Example 3: Let $M_{n\times n}$ be the semiring as in Example 1. Clearly $M_{n\times n}$ is a Smarandache semiring. Now,

$$B=\left\{\begin{pmatrix} a_{11} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & & 0 & 0 \\ 0 & 0 & \cdots & 0 & a_{mn} \end{pmatrix} \Big/ a_{11} \text{ and } a_{mn} \in Z^{+}\cup\{0\}\right\}$$

is a Smarandache sub-semiring.

Example 4: Let $M_{2\times2} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} / a,b,c,d \in Z^+ \cup \{0\}\right\}$. Clearly $M_{2\times2}$ under the matrix addition and multiplication is a semiring which is not a semifield. But $M_{2\times2}$ is a Smarandache semiring for $N = \left\{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} / a,b \in Z^+\right\} \cup \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$ is a semifield.

Theorem 4:

Not all semirings are Smarandache semirings.

Proof: Let $S = Z^+ \cup \{0\}$. $(S, +, \times)$ is a semiring which has no proper semifield contained in it. Hence the claim.

Definition 5:

The *Smarandache semifield* [4] is defined to be a semifield $(S, +, \times)$ such that a proper subset of S is a K - semi algebra (with respect with the same induced operations and an external operation).

Example 5: Let $S = Z^+ \cup \{0\}$. Now, $(S, +, \times)$ is a semifield. Consider $p \in S$, p any prime. $A = \{0, p, 2p, ...\}$ is a k-semi algebra. So $(S, +, \times)$ is a Smarandache semifield.

Consequence 1:

There also exist semifields which are not Smarandache semifields. The following example illustrates the case.

Example 6: Let $S = Q^+ \cup \{0\}$. $(S, +, \times)$ is a semifield but it is not a Smarandache semifield.

Example 7: Let $S = Z^+ \cup \{0\}$. Now $(S, +, \times)$ is a semifield. Let S[x] be polynomial semiring in the variable x. Clearly S[x] is a Smarandache semiring for S is a proper subset of S[x] is a semifield.

Theorem 5:

Let S be any semifield. Every polynomial semiring is a Smarandache semiring.

Proof: Obvious from the fact S is a semifield contained in S[x].

We now pose an open problem about the very existence of finite semirings and Smarandache semirings that are not distributive lattices.

Problem 1: Does there exist a Smarandache semiring S of finite order? (S is not a finite distributive lattice)?

Note:

We do not have finite semirings other than finite distributive lattices. Thus the existence of finite semirings other than finite distributive lattices is an open problem even in semirings.

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The sequence of prime numbers

Sebastián Martín Ruiz

9 October 2000

This article lets out a law of recurrence in order to obtain the sequence of prime numbers $\{p_k\}_{k\geq 1}$ expressing p_{k+1} as a function of $p_1, p_2, \cdots, p_k$.

Suppose we can find a function $G_k(n)$ with the following property:

$$G_k(n) = \begin{cases} -1 & if \quad n < p_{k+1} \\ 0 & if \quad n = p_{k+1} \\ something & if \quad n > p_{k+1} \end{cases}$$

This is a variation of the Smarandache Prime Function [2].

Then we can write down a recurrence formula for p_k as follows.

Consider the product:

$$\prod_{s=p_k+1}^{m} G_k(s)$$

If $p_k < m < p_{k+1}$ one has

$$\prod_{s=p_k+1}^{m} G_k(s) = \prod_{s=p_k+1}^{m} (-1) = (-1)^{m-p_k}$$

If $m \geq p_{k+1}$

$$\prod_{s=p_k+1}^{m} G_k(s) = 0$$

since $G_k(p_{k+1}) = 0$

Hence

$$\sum_{m=p_k+1}^{2p_k} (-1)^{m-p_k} \prod_{s=p_k+1}^{m} G_k(s) =$$

$$= \sum_{m=p_k+1}^{p_{k+1}-1} (-1)^{m-p_k} \prod_{s=p_k+1}^{m} G_k(s) + \sum_{m=p_{k+1}}^{2p_k} (-1)^{m-p_k} \prod_{s=p_k+1}^{m} G_k(s)$$

(The second addition is zero since all the products we have the factor $G_k(p_{k+1}) = 0$)

$$= \sum_{m=p_k+1}^{p_{k+1}-1} (-1)^{m-p_k}(-1)^{m-p_k}$$

$$= p_{k+1} - 1 - (p_k + 1) + 1 = p_{k+1} - p_k - 1$$

so

$$p_{k+1} = p_k + 1 + \sum_{m=p_k+1}^{2p_k} (-1)^{m-p_k} \prod_{s=p_k+1}^{m} G_k(s)$$

which is a recurrence relation for p_k.

We now show how to find such a function $G_k(n)$ whose definition depends only on the first k primes and not on an explicit knowledge of p_{k+1}.

And to do so we define[1]:

$$T_k(n) = \sum_{i_1=0}^{\log_{p_1} n} \sum_{i_2=0}^{\log_{p_2} n} \cdots \sum_{i_k=0}^{\log_{p_k} n} \left(\frac{n}{\prod_{s=1}^{k} p_s^{i_s}} \right)$$

Let's see the value which $T_k(n)$ takes for all $n \geq 2$ integer. We distinguish two cases:

Case 1: $n < p_{k+1}$

The expression $p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k}$ with $i_1 = 0, 1, 2 \cdots \log_{p_1} n$ $i_2 = 0, 1, 2 \cdots \log_{p_2} n$ ... $i_k = 0, 1, 2 \cdots \log_{p_k} n$ all the values occur $1, 2, 3, \cdots, n$ each one of them only once and moreover some more values, strictly greater than n.

We can look at is. If $1 \leq m \leq n$ one obtains that $m < p_{k+1}$ for which $1 \leq m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \leq n$. From where one deduces that $1 \leq p_s^{\alpha_s} \leq n$ and for it $0 \leq \alpha_s \leq \log_{p_s} n$ for all $s = 1, \cdots, k$

Therefore, for $i_s = \alpha_s \quad s = 1, 2, \cdots, k$ we have the value m. This value only appears once, the prime number descomposition of m is unique .

In fact the sums of $T_k(n)$ can be achieved up to the highest power of p_k contained in n instead of $\log_{p_k} n$.

Therefore one has that

$$T_k(n) = \sum_{i_1=0}^{\log_{p_1} n} \sum_{i_2=0}^{\log_{p_2} n} \cdots \sum_{i_k=0}^{\log_{p_k} n} \left(\frac{n}{\prod_{s=1}^{k} p_s^{i_s}} \right) = \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n - 1$$

[1]Given that $i_s \quad s = 1, 2, \cdots, k$ only takes integer values one appreciates that the sums of $T_k(n)$ are until $E(\log_{p_s} n)$ where $E(x)$ is the greatest integer less than or equal to x.

since, in the case $p_1^{i_1}p_2^{i_2}\cdots p_k^{i_k}$ would be greater than n one has that:

$$\binom{n}{\prod_{s=1}^{k} p_s^{i_s}} = 0$$

Case 2: $n = p_{k+1}$

The expression $p_1^{i_1}p_2^{i_2}\cdots p_k^{i_k}$ with $i_1 = 0,1,2\cdots\log_{p_1} n$ $i_2 = 0,1,2\cdots\log_{p_2} n$ ... $i_k = 0,1,2\cdots\log_{p_k} n$ the values occur $1,2,3,\cdots,p_{k+1}-1$ each one of them only once and moreover some more values, strictly greater than p_{k+1}.One demonstrates in a form similar to case 1. It doesn't take the value p_{k+1} since it is coprime with $p_1, p_2, \cdots, p_k$.

Therefore,

$$T_k(n) = \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} = 2^n - 2$$

In case 3: $n > p_{k+1}$ it is not necessary to consider it.

Therefore, one has:

$$T_k(n) = \begin{cases} 2^n - 1 & if \quad n < p_{k+1} \\ 2^n - 2 & if \quad n = p_{k+1} \\ something & if \quad n > p_{k+1} \end{cases}$$

and as a result:

$$G_k(n) = 2^n - 2 - T_k(n)$$

This is the summarized relation of recurrence:

Let's take $p_1 = 2$ and for $k \geq 1$ we define:

$$T_k(n) = \sum_{i_1=0}^{\log_{p_1} n} \sum_{i_2=0}^{\log_{p_2} n} \cdots \sum_{i_k=0}^{\log_{p_k} n} \binom{n}{\prod_{s=1}^{k} p_s^{i_s}}$$

$$G_k(n) = 2^n - 2 - T_k(n)$$

$$p_{k+1} = p_k + 1 + \sum_{m=p_k+1}^{2p_k} (-1)^{m-p_k} \prod_{s=p_k+1}^{m} G_k(s)$$

References:

(1) The Smarandache Notions Journal. Volume 11. Number 1-2-3. Page 59.

(2) E. Burton, "Smarandache Prime and Coprime Functions", http://www.gallup.unm.edu/~smarandache/primfnct.txt

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On a Concatenation Problem

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Abstract: This article has been inspired by questions asked by Charles Ashbacher in the *Journal of Recreational Mathematics, vol. 29.2.* It concerns the Smarandache Deconstructive Sequence. This sequence is a special case of a more general concatenation and sequencing procedure which is the subject of this study. Answers are given to the above questions. The properties of this kind of sequences are studied with particular emphasis on the divisibility of their terms by primes.

1. Introduction

In this article the concatenation of a and b is expressed by a_b or simply ab when there can be no misunderstanding. Multiple concatenations like abcabcabc will be expressed by 3(abc).

We consider n different elements (or n objects) arranged (concatenated) one after the other in the following way to form:

$A=a_1a_2\ldots a_n$.

Infinitely many objects A, which will be referred to as cycles, are concatenated to form the chain:

$B=a_1a_2\ldots a_n\,a_1a_2\ldots a_n\,a_1a_2\ldots a_n\ldots$

B contains identical elements which are at equidistant positions in the chain. Let's write B as

$B=b_1b_2b_3\ldots b_k\ldots$ where $b_k=b_j$ when $j\equiv k \pmod n$, $1\le j\le n$.

An infinte sequence $C_1, C_2, C_3, \ldots C_k, \ldots$ is formed by sequentially selecting 1,2,3, ...k, ... elements from the chain B:

$C_1=b_1=a_1$

$C2=b_2b_3=a_2a_3$

$C_3=b_4b_5b_6=a_4a_5a_6$ (if $n\le 6$, if $n=5$ we would have $C_3=a_4a_5a_1$)

The number of elements from the chain B used to form the first k-1 terms of the sequence C is 1+2+3+ ... +k-1=(k-1)k/2. Hence

$$C_k = b_{\frac{(k-1)k}{2}+1}\,b_{\frac{(k-1)k}{2}+2}\ldots b_{\frac{k(k+1)}{2}}$$

However, what is interesting to see is how C_k is expressed in terms of $a_1, \ldots a_n$. For sufficiently large values of k C_k will be composed of three parts:

The first part:	$F(k)=a_ua_{u+1}\ldots a_n$
The middle part:	M(k)=AA...A. The number of concatenated A's depends on k.
The last part:	$L(k)=a_1a_2\ldots a_w$

Hence $C_k=F(k)M(k)L(k)$ (1)

The number of elements used to form $C_1, C_2, \ldots C_k$ is (k-1)k/2. Since the number of elements in A is finite there will be infinitely many terms C_k which have the same first element a_u. u can be determined from $\frac{(k-1)k}{2}+1 \equiv u \pmod n$. There can be at most n^2 different combinations to form F(k) and L(k). Let C_j and C_i be two different terms for

which F(i)=F(j) and L(i)=L(j). They will then be separated by a number m of complete cycles of length n, i.e.

$$\frac{(j-1)j}{2}-\frac{(i-1)i}{2}=mn$$

Let's write j=i+p and see if p exists so that there is a solution for p which is independent of i.

$(i+p-1)(i+p)-(i-1)i=2mn$
$i^2+2ip+p^2-i-p-i^2+i=2mn$
$2ip+p^2-p=2mn$
$p^2+p(2i-1)=2mn$

If n is odd we will put p=n to otain n+2i-1, or m=(n+2i-1)/2. If n is even we put p=2n to obtain m=2n+2i-1. From this we see that the terms C_k have a peculiar periodic behaviour. The periodicity is p=n for odd n and p=2n for even n. Let's illustrate this for n=4 and n=5 for which the periodicity will be p=8 and p=5 respectively.

Table 1. n=4. A=abcd. B=abcd abcd abcd abcd....

i	C_i	Period #	F(i)	M(i)	L(i)
1	a		a		
2	bc		bc		
3	dab	1	d		ab
4	cdab	1	cd		ab
5	cdabc	1	cd		abc
6	dabcda	1	d	abcd	a
7	bcdabcd	1	bcd	abcd	
8	abcdabcd	1		2(abcd)	
9	abcdabcda	1		2(abcd)	a
10	bcdabcdabc	1	bcd	abcd	abc
11	dabcdabcdab	2	d	2(abcd)	ab
12	cdabcdabcdab	2	cd	2(abcd)	ab
13	cdabcdabcdabc	2	cd	2(abcd)	abc
14	dabcdabcdabcda	2	d	3(abcd)	a
15	bcdabcdabcdabcd	2	bcd	3(abcd)	
16	abcdabcdabcdabcd	2		4(abcd)	
17	abcdabcdabcdabcda	2		4(abcd)	a
18	bcdabcdabcdabcdabc	2	bcd	3(abcd)	abc
19	dabcdabcdabcdabcdab	3	d	4(abcd)	ab
20	cdabcdabcdabcdabcdab	3	cdcd	4(abcd)	ab

Note that the periodicity starts for i=3.

Numerals are chosen as elements to illustrate the case n=5. Let's write i=s+k+pj, where s is the index of the term preceding the first periodical term, k=1,2,..., p is the index of members of the period and j is the number of the period (for convenience the first period is numbered 0). The first part of C_i is denoted B(k) and the last part E(k). C_i is now given by the expression below where q is the number of cycles concatenated between the first part B(k) and the last part E(k).

$$C_i=B(k)_qA_E(k), \text{ where k is determined from } i-s\equiv k \pmod p \qquad (2)$$

Table 2. n=5. A=12345. B=123451234512345.....

i	C_i	k	q	F(i)⇔B(k)	M(i)	L(i)⇔E(k)
1	1			1		
s=2	23			2		
j=0						
3	451	1	0	45		1
4	2345	2	0	2345		
5	12345	3	1		12345	
6	123451	4	1		12345	1
7	2345123	5	0	2345		123
j=1						
3+5j	45123451	1	j	45	12345	1
4+5j	234512345	2	j	2345	12345	
5+5j	1234512345	3	j+1		2(12345)	
6+5j	12345123451	4	j+1		2(12345)	1
7+5j	234512345123	5	j	2345	12345	123
j=2						
3+5j	4512345123451	1	j	45	2(12345)	1
4+5j	23451234512345	2	j	2345	2(12345)	
....						

2. The Smarandache Deconstructive Sequence

The Smarandache Deconstructive Sequence of integers [1] is constructed by sequentially repeating the digits 1 to 9 in the following way

1,23,456,7891,23456,789123,4567891,23456789,123456789,1234567891,...

The sequence was studied in a booklet by Kashihara [2] and a number of questions on this sequence were posed by Ashbacher [3]. In thinking about these questions two observations lead to this study.

1. Why did Smarandache exclude 0 from the integers used to create the sequence? After all 0 is indispensible in all arithmetics most of which can be done using 0 and 1 only.
2. The process used to create the Deconstructive Sequence is a process which applies to any set of objects as has been shown in the introduction.

The periodicity and the general expression for terms in the "generalized deconstructive sequence" shown in the introduction may be the most important results of this study. These results will now be used to examine the questions raised by Ashbacher. It is worth noting that these divisibility questions are dealt with in base10 although only nine digits 1,2,3,4,5,6,7,8,9 are used to express terms in the sequence. In the last part of this article questions on divisibility will be posed for a deconstructive sequence generated form A="0123456789".

For i>5 (s=5) any term C_i in the sequence is composed by concatenating a first part B(k), a number q of cycles A"123456789" and a last part E(k), where i=5+k+9j, k=1,2, ... ,9, j≥0, as expressed in (2) and q=j or j+1 as shown in table 3.

Members of the Smarandache Deconstructive Sequence are now interpreted as decimal integers. The factorization of B(k) and E(k) is shown in table 3. The last two columns of this table will be useful later in this article.

Table 3. Factorization of Smarandache Deconstructive Sequence
i=5+k+9j

i	k	B(k)	q	E(k)	Digit sum	3 $\mid C_i$?
6+9j	1	789=3·263	j	123=3·41	30+j·45	3
7+9j	2	456789=3·43·3541	j	1	40+j·45	No
8+9j	3	23456789	j		44+j·45	No
9+9j	4		j+1		(j+1)-45	$9\cdot3^z$ *
10+9j	5		j+1	1	1+(j+1)-45	No
11+9j	6	23456789	j	123=3·41	50+j·45	No
12+9j	7	456789=3·43·3541	j	$123456=2^6\cdot3\cdot643$	60+j·45	3
13+9j	8	789=3·263	j+1	1	25+(j+1)-45	No
14+9j	9	23456789	j	$123456=2^6\cdot3\cdot643$	65+j·45	No

*) where z depends on j.

Together with the factoization of the cycle $A=1223456789=3^2\cdot3607\cdot3803$ it is now possible to study some divisibility properties of the sequence. We will first find a general expression for C_i in terms of j and k. For this purpose we introduce:

q(k)=0 for k=1,2,3,6,7,9 and q(k)=1 for k=4,5,8
$u(k)=1+[\log_{10}(E(k)]$ if E(k) exists otherwise u(k)=0, i.e. u(3)=u(4)=0
$\delta(j,k)=0$ if j=0 and q(k)=0 otherwise $\delta(j,k)=1$

With the help of these functions we can now use table 3 to formulate the general expression

$$C_{5+k+9j} = E(k) + \delta(j,k)\cdot A\cdot 10^{u(k)}\cdot\sum_{r=0}^{j-1+q(k)} 10^{9r} + B(k)\cdot 10^{9(j+q(k))+u(k)} \qquad (3)$$

Before dealing with the questions posed by Ashbacher we recall the familiar rules: An even number is divisible by 2; a number whose last two digits form a number which is divisible by 4 is divisible by 4. In general we have the following:

Theorem. Let N be an n-digit integer such that $N>2^\alpha$ then N is divisible by 2^α if and only if the number formed by the α last digits of N is divisible by 2^α.

Proof. To begin wwith we note that

If x divides a and x divides b then x divides (a+b).
If x divides one but not the other of a and b the x does not divide (a+b).
If neither a nor b is divisible by x then x may or may not divide (a+b).

Let's write the n-digit number in the form $a\cdot10^\alpha+b$. We then see from the following that $a\cdot10^\alpha$ is divisible by 2^α.

$10\equiv0 \pmod 2$
$100\equiv0 \pmod 4$
$1000=2^3\cdot5^3\equiv0 \pmod{2^3}$

....

$10^{\alpha} \equiv 0 \pmod{2^{\alpha}}$

and then

$a \cdot 10^{\alpha} \equiv 0 \pmod{2^{\alpha}}$ independent of a.

Now let b be the number formed by the α last digits of N, we then see from the introductory remark that N is divisible by 2^{α} if and only if the number formed by the α last digits is divisible by 2^{α}.

Question 1. Does every even element of the Smarandache Deconstructive Sequence contain at least three instances of the prime 2 as a factor?

Question 2. If we form a sequence from the elements of the Smarandache Deconstructive Sequence that end in a 6, do the powers of 2 that divide them form a monotonically increasing sequence?

These two quetions are reelated and are dealt with together. From the previous analysis we know that all even elements of the Smarandache Deconstructive end in a 6. For $i \leq 5$ they are:

$C_3 = 456 = 57 \cdot 2^3$

$C_5 = 23456 = 733 \cdot 2^5$

For $i > 5$ they are of the forms:

C_{12+9j} and C_{14+9j} which both end in ...789123456.

Examining the numbers formed by the 6, 7 and 8 last digits for divisibility by 2^6, 2^7 and 2^8 respectively we have:

$123456 = 2^6 \cdot 3 \cdot 643$

$9123456 = 2^7 \cdot 149 \cdot 4673$

89123456 is not divisible by 2^8

From this we conclude that all even Smarandache Deconstructive Sequence elements for $i \geq 12$ are divisible by 2^7 and that no elements in the sequence are divisible by higher powers of 2 than 7.

Answer to Qn 1. Yes

Answer to Qn 2. The sequence is monotonically increasing for $i \leq 12$. For $i \geq 12$ the powers of 2 that divide even elements remain constant = 2^7.

Question 3. Let x be the largest integer such that $3^x \mid i$ and y the largest integer such that $3^y \mid C_i$. It is true that x is always equal to y?

From table 3 we see that the only elements C_i of the Smarandache Deconstructive Sequence which are divisible by powers of 3 correspond to i=6+9j, 9+9j or 12+9j. Furthermore, we see that i=6+9j and C_{6+9j} are divisible by 3, no more no less. The same is true for i=12+9j and C_{12+9j}. So the statement holds in these cases. From the congruences

$9+9j \equiv 0 \pmod{3^x}$ for the index of the element

and

$45(1+j) \equiv 0 \pmod{3^y}$ for the corresponding element

we conclude that x=y.

Answer: The statement is true. It is interesting to note that, for example, the 729 digit number C_{729} is divisible by 729.

Question 4. Are there other patterns of divisibility in this sequence?

A search for patterns would continue by examining divisibility by the next lower primes 5, 7, 11, ... It is obvious from table 3 and the periodicity of the sequence that there are no elements divisible by 5. Algorithm (3) will prove useful. For each value of k the value of C_i depends on j only. The divisibility by a prime p is therefore determined by finding out for which values of j and k the congruence $C_i \equiv 0 \pmod p$ holds. We evaluate $\sum_{r=0}^{j-1+q(k)} 10^{9r} = \frac{10^{9(j+q(k))}-1}{10^9-1}$ and introduce $G=10^9-1$. We note that $G=3^4 \cdot 37 \cdot 333667$. From (3) we now obtain:

$$G \cdot C_i = G \cdot E(k) + (\delta(j,k) \cdot A \cdot + G \cdot B(k)) 10^{9(j+q(k))+u(k)} - \delta(j,k) \cdot A \cdot 10^{u(k)} \qquad (3')$$

The divisibility of C_i by a prime p other than 3, 37 and 333667 is therefore determined by solutions for j to the congruences $G \cdot C_i \equiv 0 \pmod p$ which are of the form

$$a \cdot (10^9)^j + b \equiv 0 \pmod p \qquad (4)$$

Table 4 shows the results from computer implementation of the congruences $G \cdot C_i \equiv 0 \pmod p$ for k=1,2,...9 and p<100. The appearance of elements divisible by a prime p is periodic, the periodicity is given by $j=j_1+m \cdot d$, m=1,2,3, ...The first element divisible by p appears for i_1 corresponding to j_1. In general the terms C_i divisible by p are $C_{5+k+9(j_1+md)}$ where d is specific to the prime p and m=1,2,3, ... We note from table 4 that d is either equal to p-1 or a divisor of p-1 except for the case p=37 which as we have noted is a factor of A. Indeed this periodicity follows from Euler's extension of Fermat's little theorem because we can write (mod p):

$$a \cdot (10^9)^j + b = a \cdot (10^9)^{j_1+md} + b \equiv a \cdot (10^9)^{j_1} + b \text{ for d=p-1 or a divisor of p-1.}$$

Finally we note that the periodicity for p=37 is d=37, which is found by examining (3') modulus 37^2.

Table 4. Smarandache Deconstructive elements divisible by p.

p	k	i_1	j_1	d
7	4	18	1	2
11	4	18	1	2
13	4	18	1	2
13	8	22	1	2
13	9	14	0	2
17	1	6	0	16
17	2	43	4	16
17	3	44	4	16
17	4	144	15	16
17	5	100	10	16
17	6	101	10	16
17	7	138	14	16
17	8	49	4	16
17	9	95	9	16
19	1	15	1	2
19	4	18	1	2
19	7	21	1	2
23	1	186	20	22
23	2	196	21	22
23	3	80	8	22
23	4	198	21	22
23	5	118	12	22
23	6	200	21	22
23	7	12	0	22
23	8	184	19	22
23	9	14	0	22
29	1	24	2	28
29	2	115	12	28
29	3	197	21	28
29	4	252	27	28
29	5	55	5	28
29	6	137	14	28
29	7	228	24	28
29	8	139	14	28
29	9	113	11	28
31	3	26	2	5
31	4	45	4	5
31	5	19	1	5
37	1	222	24	37
37	2	124	13	37
37	3	98	10	37
37	4	333	36	37
37	5	235	25	37
37	6	209	22	37
37	7	111	11	37
37	8	13	0	37
37	9	320	34	37
41	4	45	4	5
43	1	33	3	7
43	4	63	6	7
43	7	30	2	7

p	k	i_1	j_1	d
47	1	150	16	46
47	2	250	27	46
47	3	368	40	46
47	4	414	45	46
47	5	46	4	46
47	6	164	17	46
47	7	264	28	46
47	8	400	43	46
47	9	14	0	46
53	1	24	2	13
53	4	117	12	13
53	7	93	9	13
59	1	267	29	58
59	2	511	56	58
59	3	413	45	58
59	4	522	57	58
59	5	109	11	58
59	6	11	0	58
59	7	255	27	58
59	8	256	27	58
59	9	266	28	58
61	2	79	8	20
61	4	180	19	20
61	6	101	10	20
67	4	99	10	11
67	8	67	6	11
67	9	32	2	11
71	1	114	12	35
71	3	53	5	35
71	4	315	34	35
71	5	262	28	35
71	7	201	21	35
73	4	72	7	8
79	4	117	12	13
83	1	348	38	41
83	2	133	14	41
83	4	369	40	41
83	6	236	25	41
83	7	21	1	41
83	8	112	11	41
83	9	257	27	41
89	2	97	10	44
89	4	396	43	44
89	6	299	32	44
97	1	87	9	32
97	2	115	12	32
97	3	107	11	32
97	4	288	31	32
97	5	181	19	32
97	6	173	18	32
97	7	201	21	32
97	8	202	21	32
97	9	86	8	32

Question: Table 4 indicates some interesting patterns. For instance, the primes 19, 43 and 53 only divides elements corresponding to k=1, 4 and 7 for j<250 which was set as an upper limit for this study. Simiarly, the primes 7, 11, 41, 73 and 79 only divides elements corresponding to k=4. Is 5 the only prime that cannot divide an element of the Smarandache Deconstructive Sequence?

3. A Deconstructive Sequence generated by the cycle A=0123456789

Instead of sequentially repeating the digits 1-9 as in the case of the Smarandache Deconstructive Sequence we will use the digits 0-9 to form the corresponding sequence:

0,12,345,6789,01234,567890,1234567,89012345,678901234,5678901234, 56789012345,678901234567, ...

In this case the cycle has n=10 elements. As we have seen in the introduction the sequence then has a period 2n=20. The periodicity starts for i=8. Table 5 shows how, for i>7, any term C_i in the sequence is composed by concatenating a first part B(k), a number q of cycles A="0123456789" and a last part E(k), where i=7+k+20j, k=1,2, ... 20, j≥0, as expressed in (2) and q=2j, 2j+1 or 2j+2. In the analysis of the sequence it is important to distinguish between the cases where E(k)=0, k=6,11,14,19 and cases where E(k) does not exist, i.e. k=8,12,13,14. In order to cope with this problem we introduce a function u(k) which will at the same time replace the functions δ(j,k) and $u=1+[\log_{10}E(k)]$ used previously. u(k) is defined as shown in table 5. It is now possible to express C_i in a single formula.

$$C_i = C_{7+k+20j} = E(k) + (A \cdot \sum_{r=0}^{q(k)+2j-1} (10^{10})^r + B(k) \cdot (10^{10})^{q(k)+2j}) 10^{u(k)} \qquad (5)$$

The formula for C_i was implemented modulus prime numbers less than 100. The result is shown in table 6. Again we note that the divisibility by a prime p is periodic with a period d which is equal to p-1 or a divisor of p-1, except for p=11 and p=41 which are factors of 10^{10}-1. The cases p=3 and 5 have very simple answers and are not included in table 6.

Table 5. n=10, A=0123456789

i	k	B(k)	q	E(k)	u(k)
8+20j	1	89	2j	012345=3·5·823	6
9+20j	2	6789=3·31·73	2j	01234=2·617	5
10+20j	3	56789=109·521	2j	01234=2·617	5
11+20j	4	56789=109·521	2j	012345=3·5·823	6
12+20j	5	6789=3·31·73	2j	01234567=127·9721	8
13+20j	6	89	2j+1	0	1
14+20j	7	123456789=3^2·3607·3803	2j	01234=2·617	5
15+20j	8	56789=109·521	2j+1		0
16+20j	9		2j+1	012345=3·5·823	6
17+20j	10	6789=3·31·73	2j+1	012=2^2·3	3
18+20j	11	3456789=3·7·97·1697	2j+1	0	1
19+20j	12	123456789=3^2·3607·3803	2j+1		0
20+20j	13		2j+2		0
21+20j	14		2j+2	0	1
22+20j	15	123456789=3^2·3607·3803	2j+1	012=2^2·3	3
23+20j	16	3456789=3·7·97·1697	2j+1	012345=3·5·823	6
24+20j	17	6789=3·31·73	2j+2		0
25+20j	18		2j+2	01234=2·617	5
26+20j	19	56789=109·521	2j+2	0	1
27+20j	20	123456789=3^2·3607·3803	2j+1	01234567=127·9721	8

Table 6. Divisibility of the 10-cycle deconstructive sequence by primes p≤97

p	k	i_1	j_1	d	p	k	i_1	j_1	d
7	3	30	1	3	11	11	18	0	11
7	6	13	0	3	11	12	219	10	11
7	7	14	0	3	11	13	220	10	11
7	8	15	0	3	11	14	221	10	11
7	11	38	1	3	11	15	202	9	11
7	12	59	2	3	11	16	83	3	11
7	13	60	2	3	11	17	44	1	11
7	14	61	2	3	11	18	185	8	11
7	15	22	0	3	11	19	146	6	11
7	18	45	1	3	11	20	87	3	11
7	19	46	1	3	13	2	49	2	3
7	20	47	1	3	13	3	30	1	3
11	1	88	4	11	13	4	11	0	3
11	2	9	0	11	13	12	59	2	3
11	3	110	5	11	13	13	60	2	3
11	4	211	10	11	13	14	61	2	3
11	5	132	6	11	17	1	48	2	4
11	6	133	6	11	17	5	32	1	4
11	7	74	3	11	17	10	37	1	4
11	8	35	1	11	17	12	79	3	4
11	9	176	8	11	17	13	80	3	4
11	10	137	6	11	17	14	81	3	4

Table 6, cont. Divisibility of the 10-cycle deconstructive sequence by primes p≤97

p	k	i_1	j_1	d
17	16	43	1	4
19	1	128	6	9
19	2	149	7	9
19	3	90	4	9
19	4	31	1	9
19	5	52	2	9
19	10	117	5	9
19	12	179	8	9
19	13	180	8	9
19	14	181	8	9
19	16	63	2	9
23	1	168	8	11
23	2	149	7	11
23	3	110	5	11
23	4	71	3	11
23	5	52	2	11
23	10	217	10	11
23	12	219	10	11
23	13	220	10	11
23	14	221	10	11
23	16	223	10	11
29	2	129	6	7
29	4	11	0	7
29	10	97	4	7
29	12	139	6	7
29	13	140	6	7
29	14	141	6	7
29	16	43	1	7
31	3	30	1	3
31	9	56	2	3
31	12	59	2	3
31	13	60	2	3
31	14	61	2	3
31	17	64	2	3
37	2	9	0	3
37	3	30	1	3
37	4	51	2	3
37	12	59	2	3
37	13	60	2	3
37	14	61	2	3
41	1	788	39	41
41	2	589	29	41
41	3	410	20	41
41	4	231	11	41
41	5	32	1	41
41	6	353	17	41
41	7	614	30	41
41	8	615	30	41
41	9	436	21	41
41	10	117	5	41
41	11	678	33	41
41	12	819	40	41
41	13	820	40	41
41	14	821	40	41
41	15	142	6	41
41	16	703	34	41
41	17	384	18	41
41	18	205	9	41
41	19	206	9	41
41	20	467	22	41
43	2	109	5	21
43	3	210	10	21
43	4	311	15	21
43	6	173	8	21
43	10	217	10	21
43	12	419	20	21
43	13	420	20	21
43	14	421	20	21
43	16	203	9	21
43	20	247	11	21
47	1	28	1	23
47	2	69	3	23
47	3	230	11	23
47	4	391	19	23
47	5	432	21	23
47	6	113	5	23
47	7	214	10	23
47	8	15	0	23
47	9	376	18	23
47	12	459	22	23
47	13	460	22	23
47	14	461	22	23
47	17	84	3	23
47	18	445	21	23
47	19	246	11	23
47	20	347	16	23
53	3	130	6	13
53	12	259	12	13
53	13	260	12	13
53	14	261	12	13
59	2	269	13	29
59	3	290	14	29
59	4	311	15	29
59	7	474	23	29
59	8	395	19	29
59	9	496	24	29
59	10	297	14	29
59	11	78	3	29
59	12	579	28	29
59	13	580	28	29

Table 6, cont. Divisibility of the 10-cycle deconstructive sequence by primes p≤97

p	k	i_1	j_1	d	p	k	i_1	j_1	d
59	14	581	28	29	71	8	95	4	7
59	15	502	24	29	71	12	139	6	7
59	16	283	13	29	71	13	140	6	7
59	17	84	3	29	71	14	141	6	7
59	18	185	8	29	71	18	45	1	7
59	19	106	4	29	71	19	26	0	7
61	12	59	2	3	73	7	14	0	2
61	13	60	2	3	73	9	36	1	2
61	14	61	2	3	73	12	39	1	2
67	1	328	16	33	73	13	40	1	2
67	2	509	25	33	73	14	41	1	2
67	3	330	16	33	73	17	44	1	2
67	4	151	7	33	73	19	26	0	2
67	5	332	16	33	79	1	228	11	13
67	6	273	13	33	79	3	130	6	13
67	7	234	11	33	79	5	32	1	13
67	8	95	4	33	79	12	259	12	13
67	9	56	2	33	79	13	260	12	13
67	10	557	27	33	79	14	261	12	13
67	11	378	18	33	83	3	410	20	41
67	12	659	32	33	83	9	476	23	41
67	13	660	32	33	83	12	819	40	41
67	14	661	32	33	83	13	820	40	41
67	15	282	13	33	83	14	821	40	41
67	16	103	4	33	83	17	344	16	41
67	17	604	29	33	89	12	219	10	11
67	18	565	27	33	89	13	220	10	11
67	19	426	20	33	89	14	221	10	11
67	20	387	18	33	97	8	455	22	24
71	1	8	0	7	97	12	479	23	24
71	3	70	3	7	97	13	480	23	24
71	5	132	6	7	97	14	481	23	24
71	7	114	5	7	97	18	25	0	24

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On a Deconcatenation Problem

Henry Ibstedt

Abstract: In a recent study of the *Primality of the Smarandache Symmetric Sequences* Sabin and Tatiana Tabirca [1] observed a very high frequency of the prime factor 333667 in the factorization of the terms of the second order sequence. The question if this prime factor occurs peridically was raised. The odd behaviour of this and a few other primefactors of this sequence will be explained and details of the periodic occurence of this and of several other prime factors will be given.

Definition: The n*th* term of the Smarandache symmetric sequence of the second order is defined by S(n)=123...n_n...321 which is to be understood as a concatenation[1] of the first n natural numbers concatenated with a concatenation in reverse order of the n first natural numbers.

Factorization and Patterns of Divisibility

The first five terms of the sequence are: 11, 1221, 123321, 12344321, 1234554321. The number of digits D(n) of S(n) is growing rapidly. It can be found from the formula:

$$D(n) = 2k(n+1) - \frac{2(10^k - 1)}{9} \text{ for n in the interval } 10^{k-1} \le n < 10^k - 1 \qquad (1)$$

In order to study the repeated occurrance of certain prime factors the table of S(n) for $n \le 100$ produced in [1] has been extended to $n \le 200$. Tabirca's aim was to factorize the terms S(n) as far as possible which is more ambitious then the aim of the present calculation which is to find prime factors which are less than 10^8. The result is shown in table 1.

The computer file containing table 1 is analysed in various ways. Of the 664579 primes which are smaller than 10^7 only 192 occur in the prime factoriztions of S(n) for $1 \le n \le 200$. Of these 192 primes 37 occur more than once. The record holder is 333667, the 28693th prime, which occurs 45 times for $1 \le n \le 200$ while its neighbours 333647 and 333673 do not even occur once. Obviously there is something to be explained here. The frequency of the most frequently occurring primes is shown below..

Table 2. Most frequently occurring primes

p	3	33367	37	41	271	9091	11	43	73	53	97	31	47
Freq	132	45	41	41	41	29	25	24	14	8	7	6	6

[1] In this article the concatenation of a and b is written a_b. Multiplication ab is often made explicit by writing a.b. When there is no reason for misunderstanding the signs "_" and "." are omitted. Several tables contain prime factorizations. Prime factors are given in ascending order, multiplication is expressed by "." and the last factor is followed by ".." if the factorization is incomplete or by Fxxx indicating the number of digits of the last factor. To avoid typing errors all tables are electronically transferred from the calculation program, which is DOS-based, to the wordprocessor. All editing has been done either with a spreadsheet program or directly with the text editor. Full page tables have been placed at the end of the article. A non-proportional font has been used to illustrate the placement of digits when this has been found useful.

The distribution of the primes 11, 37, 41, 43, 271, 9091 and 333667 is shown in table 3. It is seen that the occurance patterns are different in the intervals $1 \le n \le 9$, $10 \le n \le 99$ and $100 \le n \le 200$. Indeed the last interval is part of the interval $100 \le n \le 999$. It would have been very interesting to include part of the interval $1000 \le n \le 9999$ but as we can see from (1) already S(1000) has 5786 digits. Partition lines are drawn in the table to highlight the different intervals. The less frequent primes are listed in table 4 where primes occurring more than once are partitioned.

From the patterns in table 3 we can formulate the occurance of these primes in the intervals $1 \le n \le 9$, $10 \le n \le 99$ and $100 \le n \le 200$, where the formulas for the last interval are indicative. We note, for example, that 11 is not a factor of any term in the interval $100 \le n \le 999$. This indicates that the divisibility patterns for the interval $1000 \le n \le 9999$ and further intervals is a completely open question.

Table 5 shows an analysis of the patterns of occurance of the primes in table 1 by interval. Note that we only have observations up to n=200. Nevertheless the interval $100 \le n \le 999$ is used. This will be justified in the further analysis.

Table 5. Divisibility patterns

Interval	p	n	Range for j
$1 \le n \le _$	3	2+3j	j=0,1,_
$1 \le n \le _$		3j	j=1,2,_
$1 \le n \le 9$	11	All values of n	
$10 \le n \le 99$		12+11j	j=0,1, _ ,7
		20+11j	j=0,1, _ ,7
$100 \le n \le 999$		None	
$1 \le n \le 9$	37	2+3j	j=0,1,2
		3+3j	j=0,1,2
$10 \le n \le 99$		12+3j	j=0,1,_,28,29
$100 \le n \le 999$		122+37j	j=0,1,_,23
		136+37j	j=0,1,_,23
$1 \le n \le 9$	41	4+5j	j=0,1
		5	
$10 \le n \le 999$		14+5j	j=0,1,_,197
$1 \le n \le 9$	43	None	
$10 \le n \le 99$		11+21j	j=0,1,3,4
		24+21j	j=0,1,2,3
$100 \le n \le 999$		100	
		107+7j	j=0,1,_,127
$1 \le n \le 9$	271	4+5j	j=0,1
		5	
$10 \le n \le 999$		14+5j	j=0,1,_,197
$1 \le n \le 999$	9091	9+5j	j=0,1,_,98
$1 \le n \le 9$	333667	8,9	
$10 \le n \le 99$		18+9j	j=0,1,_,9
$100 \le n \le 999$		102+3j	j=0,1,_,299

We note that no terms are divisible by 11 for $n>100$ in the interval $100 \le n \le 200$ and that no term is divisible by 43 in the interval $1 \le n \le 9$. Another remarkable observation is that the sequence shows exactly the same behaviour for the primes 41 and 271 in the intervals included in the study. Will they show the same behaviour when $n \ge 1000$?

Consider

$S(n)=12\ldots n_n\ldots 21.$

Let p be a divisor of S(n). We will construct a number

$N=12\ldots n_0..0_n\ldots 21$ (2)

so that p also divides N. What will be the number of zeros? Before discussing this let's consider the case p=3.

Case 1. p=3.

In the case p=3 we use the familiar rule that a number is divisible by 3 if and only if its digit sum is divisible by 3. In this case we can insert as many zeros as we like in (2) since this does not change the sum of digits. We also note that any integer formed by concatenation of three consecutive integers is divisible by 3, cf a_a+1_a+2, digit sum 3a+3. It follows that also a_a+1_a+2_a+2_a+1_a is divisible by 3. For a=n+1 we insert this instead of the appropriate number of zeros in (2). This means that if $S(n)\equiv 0 \pmod 3$ then $S(n+3)\equiv 0 \pmod 3$. We have seen that $S(2)\equiv 0 \pmod 3$ and $S(3)\equiv 0 \pmod 3$. By induction it follows that $S(2+3j)\equiv 0 \pmod 3$ for j=1,2,... and $S(3j)\equiv 0 \pmod 3$ for j=1,2,... .

We now return to the general case. S(n) is deconcatenated into two numbers 12...n and n... 21 from which we form the numbers

$A=12\ldots n\cdot 10^{1+[\log_{10} B]}$ and B=n...21

We note that this is a different way of writing S(n) since indeed A+B=S(n) and that $A+B\equiv 0 \pmod p$. We now form $M=A\cdot 10^s+B$ where we want to determine s so that $M\equiv 0 \pmod p$. We write M in the form $M=A(10^s-1)+A+B$ where A+B can be ignored mod p. We exclude the possibility $A\equiv 0 \pmod p$ which is not interesting. This leaves us with the congruence

$M\equiv A(10^s-1)\equiv 0 \pmod p$

or

$10^s-1\equiv 0 \pmod p$

We are particularly interested in solutions for which

$p\in\{11,37,41,43,271,9091,333667\}$

By the nature of the problem these solutions are periodic. Only the two first values of s are given for each prime.

Table 6. $10^s-1\equiv 0 \pmod p$

p	3	11	37	41	43	271	9091	33367
s	1,2	2,4	3,6	5,10	21,42	5,10	10,20	9,18

We note that the result is independent of n. This means that we can use n as a parameter when searching for a sequence C=n+1_n+2_...n+k_n+k_...n+2_n+1 such that this is also divisible by p and hence can be inserted in place of the zeros to form S(n+k) which then fills the condition $S(n+k)\equiv 0 \pmod p$. Here k is a multiple of s or s/2 in case s is even. This explains the results which we have already obtained in a different way as part of the factorization of S(n) for $n\leq 200$, see tables 3 and 5. It remains to explain the periodicity which as we have seen is different in different intervals $10^u\leq n\leq 10^u-1$.

This may be best done by using concrete examples. Let us use the sequences starting with n=12 for p=37, n=12 and n=20 for p=11 and n=102 for p=333667. At the same time we will illustrate what we have done above.

Case 2: n=12, p=37. Period=3. Interval: 10≤n≤99.

```
S(n)=  123456789101112_____________121110987654321
N=     12345678910111200000000000001211109876543321
C=                    131415151413
S(n+k)=1234567891011121314151514131211109876543321
```

Let's look at C which carries the explanation to the periodicity. We write C in the form

$C=101010101010+30405050403$

We know that C≡0 (mod 37). What about 101010101010? Let's write

$101010101010=10+10^3+10^5+\ldots+10^{11}=(10^{12}-1)/9\equiv 0 \pmod{37}$

This congruence mod 37 has already been established in table 6. It follows that also

$30405050403\equiv 0 \pmod{37}$

and that

$x\cdot(101010101010)\equiv 0 \pmod{37}$ for x = any integer

Combining these observations we se that

$232425252423,\ 333435353433,\ \ldots\ 939495959493\equiv 0 \pmod{37}$

Hence the periodicity is explained.

Case 3a: n=12, p=11. Period=11. Interval: 10≤n≤99.

```
S(12)=12_.._12_____________________________________________12_.._21
S(23)=12_.._121314151617181920212223232221201918171615141312_.._21
C=             131415161718192021222323222120191817161514 13=
C1=            101010101010101010101010101010101010101010+
C2=             304050607080910111213131211100908070605040 3
```

From this we form

```
2·C1+C2=        2324252627282930313233333231302928272625242 3
```

which is NOT what we wanted, but C1≡0 (mod11) and also C1/10≡0 (mod 11). Hence we form

$2\cdot C1+C1/10+C2=2425262728293031323334343332313029282726252 4$

which is exactly the C-term required to form the next term S(34) of the sequence. For the next term S(45) the C-term is formed by 3·C1+2·C1/10+C2 The process is repeated adding C1+C1/10 to proceed from a C-term to the next until the last term <100, i.e. S(89) is reached.

Case 3b: n=20, p=11. Period=11. Interval: 10≤n≤99.

This case does not differ much from the case n=12. We have

```
S(20)=12_.._20_____________________________________________20_.._21
S(31)=12_.._2021222324252627282930313130292827262524232221 20_.._21
C=             212223242526272829303131302928272625242322212 0=
C1=            101010101010101010101010101010101010101010101+
C2=             102030405060708091011111009080706050403020 1
```

The C-term for S(42) is

$3\cdot C1+C1/10+C2=32333435363738394041424241403938373635343332$

In general C=x·C1+(x-1)·C1/10+C2 for x=3,4,5, ..,8. For x=8 the last term S(97) of this sequence is reached.

Case 4: n=102, p=333667. Period=3. Interval: 100≤n≤999.

```
S(102)=12_.._101102_____________________102101_.._21
S(105)=12_.._101102103104105105104103102101_.._21
C=                  103104105105104103              ≡0 (mod 333667)
C1=                 100100100100100100              ≡0 (mod 333667)
C2=                   3004005005004003              ≡0 (mod 333667)
```

Removing 1 or 2 zeros at the end of C1 does not affect the congruence modulus 333667, we have:

```
C1'=                 10010010010010010              ≡0 (mod 333667)
C1''=                 1001001001001001              ≡0 (mod 333667)
```

We now form the combinations:

$$x\cdot C1+y\cdot C1'+z\cdot C1''+C2\equiv 0 \pmod{333667}$$

This, in my mind, is quite remarkable: All 18-digit integers formed by the concatenation of three consecutive 3-digit integers followed by a concatenation of the same integers in descending order are divisible by 333667, example $376377378378377376\equiv 0 \pmod{333667}$. As far as the C-terms are concerned all S(n) in the range $100\leq n\leq 999$ could be divisible by 333667, but they are not. Why? It is because S(100) and S(101) are not divisible by 333667. Consequently n=100+3k and 101+3k can not be used for insertion of an appropriate C-value as we did in the case of S(102). This completes the explanation of the remarkable fact that every third term S(102+3j) in the range $100\leq n\leq 999$ is divisible by 333667.

These three cases have shown what causes the periodicity of the divisibility of the Smarandache symmetric sequence of the second order by primes. The mechanism is the same for the other periodic sequences.

Beyond 1000

We have seen that numbers of the type:

```
10101010…10, 100100100…100, 10001000…1000, etc
```

play an important role. Such numbers have been factorized and the occurrence of our favorite primes 11, 37, ..., 333667 have been listed in table 7. In this table a number like 100100100100 has been abbreviated 4(100) or q(E), where q and E are listed in separate columns.

Question 1. Does the sequence of terms S(n) divisible by 333667 continue beyond 1000?

Although S(n) was partially factorized only up n=200 we have been able to draw conclusions on divisibility up n=1000. The last term that we have found divisible by 333667 is S(999). Two conditions must be met for there to be a sequence of terms divisible by p=333667 in the interval $1000\leq n\leq 9999$.

Condition 1. There must exist a number 10001000...1000 divisible by 333667 to ensure the periodicity as we have seen in our case studies.
In table 7 we find q=9, E=1000. This means that the periodicity will be 9 – if it exists, i.e. condition 1 is met.

Condition 2. There must exist a term S(n) with n≥1000 divisible by 333667 which will constitute the first term of the sequence.
The last term for n<1000 which is divisible by 333667 is S(999) from which we build

S(108)=12_999_1000_..._1008_1008_..1000_999-_21

where we deconcatenate 100010011002...10081008...10011000 which is divisible by 333667 and provides the C-term (as introduced in the case studies) needed to generate the sequence, i.e. condition 2 is met.

We conclude that S(1008+9j)≡0 (mod 333667) for j=0,1,2, ... 999. The last term in this sequence is S(9999). From table 7 we see that there could be a sequence with the period 9 in the interval 10000≤n≤99999 and a sequence with period 3 in the interval 100000≤n≤999999. It is not difficult to verify that the above conditions are filled also in these intervals. This means that we have:

S(1008+9j)≡0 (mod 333667)	for j=01,2,...,999, i.e. $10^3 \le n \le 10^4-1$
S(10008+9j)≡0 (mod 333667)	for j=01,2,...,9999, i.e. $10^4 \le n \le 10^5-1$
S(100002+3j)≡0 (mod 333667)	for j=01,2,...,99999, i.e. $10^5 \le n \le 10^6-1$

It is one of the fascinations with large numbers to find such properties. This extraordinary property of the prime 333667 in relation to the Smarandache symmetric sequence probably holds for $n>10^6$. It easy to loose contact with reality when plying with numbers like this. We have S(999999)≡0 (mod 333667). What does this number S(999999) look like? Applying (1) we find that the number of digits D(999999) of S(999999) is

$$D(999999)=2\cdot 6\cdot 10^6-2\cdot(10^6-)/9=11777778$$

Let's write this number with 80 digits per line, 60 lines per page, using both sides of the paper. We will need 1226 sheets of paper – more that 2 reams!

Question 2. Why is there no sequence of S(n) divisible by 11 in the interval 100≤n≤999?

Condition1. We must have a sequence of the form 100100.. divisible by 11 to ensure the periodicity. As we can see from table 7 the sequence 100100 fills the condition and we would have a periodicity equal to 2 if the next condition is met.

Condition 2. There must exist a term S(n) with n≥100 divisible by 11 which would constitute the first term of the sequence. This time let's use a nice property of the prime 11:

$$10^s\equiv(-1)^s \pmod{11}$$

Let's deconcatenate the number a_b corresponding to the concatenation of the numbers a and b: We have:

$$a_b=a\cdot 10^{1+[\log_{10} b]}+b=\begin{cases} -a+b \text{ if } 1+[\log_{10}b] \text{ is odd} \\ a+b \text{ if } 1+[\log_{10}b] \text{ is even} \end{cases}$$

Let's first consider a deconcatenated middle part of S(n) where the concatenation is done with three-digit integers. For convienience I have chosen a concrete example – the generalization should pose no problem

```
27327427527527427 3≡2-7+3-2+7-4+2-7+5-2+7-5+2-7+4-2+7-3≡0 (mod 11)
                  +-+-+-+-+-+-+-+-+-+-
```

It is easy to see that this property holds independent of the length of the sequence above and whether it start on + or -. It is also easy to understand that equivalent results are obtained for other primes although factors other than +1 and –1 will enter into the picture.

We now return to the question of finding the first term of the sequence. We must start from n=97 since S(97) it the last term for which we know that S(n)≡0 (mod 11). We form:

```
9899100101_n_n_1011009998≡2 (mod 11) independent of n<1000.
+-+-+-+-+-_ _ _-+-+-+-+-+-
```

This means that S(n)≡2 (mod 11) for 100≤n≤999 and explains why there is no sequence divisible by 11 in this interval.

Question 3. Will there be a sequence divisible by 11 in the interval 1000≤n≤9999?

Condition 1. A sequence 10001000...1000 divisible by 11 exists and would provide a period of 11, se table 7.

Condition 2. We need to find one value n ≥1000 for which S(n)≡0 (mod 11). We have seen that S(999)≡2 (mod 11). We now look at the sequences following S(999). Since S(999)≡2 (mod 9) we need to insert a sequence 10001001..m_m...10011000≡9 (mod 11) so that S(m)≡0 (mod 11). Unfortunately m does not exist as we will see below

```
10001000≡2 (mod 11)
+-+-+-+-
1   1
1000100110011000≡2. (mod 11)
+-+-+-+-+-+-+-+-
1   1   1   1
      1   1
1000100110021002100110000≡0 (mod 11)
+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-
1   1   1   1   1   1
      1   2   2   1
1000100110021003100310021001100 0≡-4≡7 (mod 11)
+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+
1   1   1   1   1   1   1   1
      1   2   3   3   2   1
```

Continuing this way we find that the residues form the period 2,2,0,7,1,4,5,4,1,7,0. We needed a residue to be 9 in order to build sequences divisible by 9. We conclude that S(n) is not divisible by 11 in the interval 1000≤n≤9999.

Trying to do the above analysis with the computer programs used in the early part of this study causes overflow because the large integers involved. However, changing the approach and performing calculations modulus 11 posed no problems. The above method was preferred for clarity of presentation.

Epilog

There are many other questions that may be interesting to look into. This is left to the reader. The author's main interest in this has been to develop means by which it is possible to identify some properties of large numbers other than the so frequently asked question as to whether a big number is a prime or not. There are two important ways to generate large numbers that I found particularly interesting – iteration and concatenation. In this article the author has drawn on work done previously, references below. In both these areas very large numbers may be generated for which it may be impossible to find any practical use – the methods are often more important than the results.

References:

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Table 1. Prime factors of S(n) which are less than 10^8

n	Prime factors of S(n)	n	Prime factors of S(n)
1	11	51	3.37.1847.F180
2	3.11.37	52	F190
3	3.11.37.101	53	3^3.11.43.26539.17341993.F178
4	11.41.101.271	54	3^3.37.41.151.271.347.463.9091.333667.F174
5	3.7.11.13.37.41.271	55	67.F200
6	3.7.11.13.37.239.4649	56	3.11.F204
7	11.73.101.137.239.4649	57	3.31.37.F206
8	3^2.11.37.73.101.137.333667	58	227.9007.20903089.F200
9	3^2.11.37.41.271.9091.333667	59	3.41.97.271.9091.F207
10	F22	60	3.37.3368803.F213
11	3.43.97.548687.F16	61	91719497.F218
12	3.11.31.37.61.92869187.F15	62	3^2.1693.F225
13	109.3391.3631.F24	63	3^2.37.305603.333667.9136499.F213
14	3.41.271.9091.290971.F24	64	11.41.271.9091.F229
15	3.37.661.F37	65	3.839.F238
16	F46	66	3.37.43.F242
17	3.F49	67	11^2.109.467.3023.4755497.F233
18	3^2.37.1301.333667.6038161.87958883.F28	68	3.97.5843.F247
19	41.271.9091.F50	69	3.37.41.271.787.9091.716549.19208653.F232
20	3.11.97.128819.F53	70	F262
21	3.37.983.F61	71	3.F265
22	67.773.F65	72	3^2.31.37.61.163.333667.77696693.F248
23	3.11.7691.F68	73	379.323201.F266
24	3.37.41.43.271.9091.165857.F61	74	$3.41^2.43^2$.179.271.9091.8912921.F255
25	227.2287.33871.611999.F66	75	3.11.37.443.F276
26	3^3.163.5711.68432503.F70	76	1109.F283
27	3^3.31.37.333667.481549.F74	77	3.10034243.F282
28	146273.608521.F83	78	3.11.37.71.41549.F284
29	3.41.271.9091.F89	79	41.271.9091.F290
30	3.37.5167.F96	80	3.F300
31	11^3.4673.F99	81	3^5.37.333667.4274969.F289
32	3.43.1021.F104	82	F310
33	3.37.881.F109	83	3.20399.5433473.F302
34	11.41.271.9091.F109	84	3.37^2.41.271.9091.F306
35	3^2.3209.F117	85	1783.627041.F313
36	3^2.37.333667.68697367.F110	86	3.11.F324
37	F130	87	3.31.37.43.F324
38	3.1913.12007.58417.597269.63800419.F107	88	67.257.46229.F325
39	3.37.41.271.347.9091.23473.F121	89	3^2.11.41.271.9091.653659.76310887.F314
40	F142	90	3^2.37.244861.333667.F328
41	3.156841.F140	91	173.F343
42	3.11.31.37.61.20070529.F136	92	3.F349
43	71.5087.F148	93	3.37.1637.F348
44	3^2.41.271.9091.1553479.F142	94	41.271.9091.10671481.F343
45	3^2.11.37.43.333667.F151	95	3.43.2833.F356
46	F166	96	3.37.683.F361
47	3.F169	97	11.26974499.F361
48	3.37.173.60373.F165	98	3^2.1299169.F367
49	41.271.929.9091.34613.F162	99	3^2.37.41.271.2767.9091.263273.333667.481417.F347
50	3.167.1789.9923.F172	100	43.47.53.83.683.3533.4919.F367

Table 1 continued

n	Prime factors of S(n)	n	Prime factors of S(n)
101	3.F389	151	47.5783.405869.F679
102	3.149.21613.106949.333667.F378	152	3^2.53.F693
103	45823.F397	153	3^2.359.39623.333667.7192681.F681
104	3.41.271.28813.F399	154	41.73.271.487.14843.F695
105	3.47.333667.11046661.F399	155	3.14717.F709
106	73.167.F416	156	3.43.601.1289.14153.333667.1479589.11337C 23.F689
107	3^3.43.1447.1741.28649.161039.F406	157	F726
108	3^3.569.333667.F422	158	3.49055933.F723
109	41.271.367.9091.F427	159	3.37.41.271.347.9091.333667.F719
110	3.F443	160	97.179.1277.F736
111	3.313.333667.F441	161	3^4.3251.75193.496283.F734
112	F456	162	3^4.73.26881.28723.333667.3211357.F731
113	3.53.71.2617.52081.F449	163	43.1663.F757
114	3.41.43.73.271.333667.F454	164	3.41.271.136319.F758
115	2309.F470	165	3.53.83.919.184859.333667.3014983.F749
116	3.F479	166	1367.1454371.F770
117	3^2.333667.4975757.F472	167	3.F785
118	167.11243.13457.414367.F476	168	3.19913.333667.F781
119	3.41.271.9091.132059.182657.F479	169	41.271.2273.9091.F786
120	3.1511.7351.20431.167611.333667.572282 99.F473	170	3^2.43.73.967.F796
121	43.501233.F502	171	3^2.333667.F803
122	3.37.73.2659.F508	172	643.96293.325681.7607669.F795
123	3.112207.333667.F511	173	3.37.F820
124	41.83.271.367.37441.F514	174	3.41.271.19423.333667.F813
125	3.F533	175	3607.20131291.F823
126	3^2.53.333667.395107.972347.F520	176	3.F839
127	F546	177	3.43.173.333667.F836
128	3.43.97.179.181.347.F540	178	53.73.11527.461317.F838
129	3.41.271.9091.333667.F544	179	3^2.41.271.1033.9091.F846
130	73.313.275083.F554	180	3^2.2861.26267.333667.1894601.F843
131	3.263.12511.210491.95558129.F549	181	F870
132	3.333667.F570	182	3.83.2417.F870
133	F582	183	3.71.1097.333667.F871
134	3^3.41.173.271.F580	184	41.43.271.F882
135	3^3.43.59.333667.F583	185	3.317371.F888
136	37.F598	186	3.73.333667.F892
137	3.F605	187	F906
138	3.73.28817.333667.F599	188	3^3.181.1129.5179.F901
139	41.53.271.9091.19433.F604	189	3^3.41.271.9091.13627.333667.F898
140	3.380623.F618	190	194087.F918
141	3.83.257.1091.333667.29618101.F609	191	3.43.53.401.F923
142	43.F634	192	3.47.97.333667.14445391.F919
143	3^2.8922281.F634	193	59.F940
144	3^2.41.59.271.1493.333667.F632	194	3.41.73.271.487.42643.F934
145	977.22811.5199703.F640	195	3.179533.333667.F942
146	3.47.73.F656	196	37.661.F955
147	3.1483.2341.333667.F653	197	3^2.47.18427.6309143.32954969.F944
148	71.14271083.47655077.F655	198	3^2.43^2.333667.F962
149	3.41.43.271.9091.F667	199	41.271.9091.10151.719779.F960
150	3.333667.F678	200	3.4409.F979

Table 3. Smarandache Symmetric Sequence of Second Order: The most frequently occurring prime factors.

#	11	diff	#	37	diff	#	41	diff	#	43	diff	#	271	diff	#	9091	diff	#	333667	diff
1	11		2	37		4	41		11	43		4	271		9	9091		8	333667	
2	11	1	3	37	1	5	41	1	24	43	13	5	271	1	14	9091	5	9	333667	1
3	11	1	5	37	2	9	41	4	32	43	8	9	271	4	19	9091	5	18	333667	9
4	11	1	6	37	1	14	41	5	45	43	13	14	271	5	24	9091	5	27	333667	9
5	11	1	8	37	2	19	41	5	53	43	8	19	271	5	29	9091	5	36	333667	9
6	11	1	9	37	1	24	41	5	66	43	13	24	271	5	34	9091	5	45	333667	9
7	11	1	12	37	3	29	41	5	74	43	8	29	271	5	39	9091	5	54	333667	9
8	11	1	15	37	3	34	41	5	87	43	13	34	271	5	44	9091	5	63	333667	9
9	11	1	18	37	3	39	41	5	95	43	8	39	271	5	49	9091	5	72	333667	9
12	11	3	21	37	3	44	41	5	100	43	5	44	271	5	54	9091	5	81	333667	9
20	11	8	24	37	3	49	41	5	107	43	7	49	271	5	59	9091	5	90	333667	9
23	11	3	27	37	3	54	41	5	114	43	7	54	271	5	64	9091	5	99	333667	9
31	11	8	30	37	3	59	41	5	121	43	7	59	271	5	69	9091	5	102	333667	3
34	11	3	33	37	3	64	41	5	128	43	7	64	271	5	74	9091	5	105	333667	3
42	11	8	36	37	3	69	41	5	135	43	7	69	271	5	79	9091	5	108	333667	3
45	11	3	39	37	3	74	41	5	142	43	7	74	271	5	84	9091	5	111	333667	3
53	11	8	42	37	3	79	41	5	149	43	7	79	271	5	89	9091	5	114	333667	3
56	11	3	45	37	3	84	41	5	156	43	7	84	271	5	94	9091	5	117	333667	3
64	11	8	48	37	3	89	41	5	163	43	7	89	271	5	99	9091	5	120	333667	3
67	11	3	51	37	3	94	41	5	170	43	7	94	271	5	109	9091	10	123	333667	3
75	11	8	54	37	3	99	41	5	177	43	7	99	271	5	119	9091	10	126	333667	3
78	11	3	57	37	3	104	41	5	184	43	7	104	271	5	129	9091	10	129	333667	3
86	11	8	60	37	3	109	41	5	191	43	7	109	271	5	139	9091	10	132	333667	3
89	11	3	63	37	3	114	41	5	198	43	7	114	271	5	149	9091	10	135	333667	3
97	11	8	66	37	3	119	41	5				119	271	5	159	9091	10	138	333667	3
			69	37	3	124	41	5				124	271	5	169	9091	10	141	333667	3
			72	37	3	129	41	5				129	271	5	179	9091	10	144	333667	3
			75	37	3	134	41	5				134	271	5	189	9091	10	147	333667	3
			78	37	3	139	41	5				139	271	5	199	9091	10	150	333667	3
			81	37	3	144	41	5				144	271	5				153	333667	3
			84	37	3	149	41	5				149	271	5				156	333667	3
			87	37	3	154	41	5				154	271	5				159	333667	3
			90	37	3	159	41	5				159	271	5				162	333667	3
			93	37	3	164	41	5				164	271	5				165	333667	3
			96	37	3	169	41	5				169	271	5				168	333667	3
			99	37	3	174	41	5				174	271	5				171	333667	3
			122	37	23	179	41	5				179	271	5				174	333667	3
			136	37	14	184	41	5				184	271	5				177	333667	3
			159	37	23	189	41	5				189	271	5				180	333667	3
			173	37	14	194	41	5				194	271	5				183	333667	3
			196	37	23	199	41	5				199	271	5				186	333667	3
																		189	333667	3
																		192	333667	3
																		195	333667	3
																		198	333667	3

Table 4. Smarandache Symmetric Sequence of Second Order: Less frequently occurring prime factors.

#	p	d	#	p	d	#	p	d	#	p	d	#	p	d	#	p	d	#	p
5	7		7	73		50	167		15	661		147	2341		154	14843		24	165857
6	7	1	8	73	1	106	167	56	196	661		182	2417		197	18427		120	167611
5	13		106	73	98	118	167	12	96	683		113	2617		174	19423		195	179533
6	13	1	114	73	8	48	173		100	683		122	2659		139	19433		119	182657
12	31		122	73	8	91	173	43	22	773		99	2767		168	19913		165	184859
27	31	15	130	73	8	134	173	43	69	787		95	2833		83	20399		190	194087
42	31	15	138	73	8	177	173	43	65	839		180	2861		120	20431		131	210491
57	31	15	146	73	8	74	179		33	881		67	3023		102	21613		90	244861
72	31	15	154	73	8	128	179	54	165	919		35	3209		145	22811		99	263273
87	31	15	162	73	8	160	179	32	49	929		161	3251		39	23473		130	275083
100	47		170	73	8	128	181		170	967		13	3391		180	26267		14	290971
105	47	5	178	73	8	188	181		145	977		100	3533		53	26539		63	305603
146	47	41	186	73	8	25	227		21	983		175	3607		162	26881		185	317371
151	47	5	194	73	8	58	227		32	1021		13	3631		107	28649		73	323201
192	47	41	100	83		6	239		179	1033		200	4409		162	28723		172	325681
197	47	5	124	83	24	7	239		141	1091		6	4649		104	28813		140	380623
100	53		141	83	17	88	257		183	1097		7	4649		138	28817		126	395107
113	53	13	165	83	24	141	257		76	1109		31	4673		25	33871		151	405869
126	53	13	182	83	17	131	263		188	1129		100	4919		49	34613		118	414367
139	53	13	11	97		111	313		160	1277		43	5087		124	37441		178	461317
152	53	13	20	97	9	130	313		156	1289		30	5167		153	39623		99	481417
165	53	13	59	97	39	39	347		18	1301		188	5179		78	41549		27	481549
178	53	13	68	97	9	54	347	15	166	1367		26	5711		194	42643		161	496283
191	53	13	128	97	60	128	347	74	107	1447		151	5783		103	45823		121	501233
135	59		160	97	32	159	347	31	147	1483		68	5843		88	46229		11	548687
144	59	9	192	97	32	153	359		144	1493		120	7351		113	52081		38	597269
193	59	49	3	101		109	367		120	1511		23	7691		38	58417		28	608521
12	61		4	101	1	124	367		93	1637		58	9007		48	60373		25	611999
42	61	30	7	101	3	73	379		163	1663		50	9923		161	75193		85	627041
72	61	30	8	101	1	191	401		62	1693		199	10151		172	96293		89	653659
22	67		13	109		75	443		107	1741		118	11243		102	106949		69	716549
55	67	33	67	109		54	463		85	1783		178	11527		123	112207		199	719779
88	67	33	7	137		67	467		50	1789		38	12007		20	128819		126	972347
43	71		8	137		154	487		51	1847		131	12511		119	132059			
78	71	35	102	149		194	487		38	1913		118	13457		164	136319			
113	71	35	54	151		108	569		169	2273		189	13627		28	146273			
148	71	35	26	163		156	601		25	2287		156	14153		41	156841			
183	71	35	72	163		172	643		115	2309		155	14717		107	161039			

Table 7. Prime factors of q(E) and occurrence of selected primes

q	E	Prime factors <350000	Selected primes
2	10	2.5.101	
3	10	2.3.5.7.13.37	37
4	10	2.5.73.101.137	
5	10	2.5.41.271.9091	41,271,9091
6	10	2.3.5.7.13.37.101.9901	37,9091
7	10	2.5.239.4649.	
8	10	2.5.17.73.101.137.	
9	10	2.3^2.5.7.13.19.37.52579.333667	333667
10	10	2.5.41.101.271.3541.9091.27961	41,271,9091
11	10	2.5.11.23.4093.8779.21649.	11
12	10	2.3.5.7.13.37.73.101.137.9901.	37
13	10	2.5.53.79.859.	
14	10	2.5.29.101.239.281.4649.	
15	10	2.3.5.7.13.31.37.41.211.241.271.2161.9091.	37,41,271,9091
16	10	2.5.17.73.101.137.353.449.641.1409.69857.	
2	100	$2^2.5^2$.7.11.13	11
3	100	$2^2.5^2$.333667	333667
4	100	$2^2.5^2$.7.11.13.101.9901	11
5	100	$2^2.5^2$.31.41.271.	41,271
6	100	$2^2.3.5^2$.7.11.13.19.52579.333667	11,333667
7	100	$2^2.5^2$.43.239.1933.4649.	43
8	100	$2^2.5^2$.7.11.13.73.101.137.9901.	11,73
9	100	$2^2.3^2.5^2$.757.333667.	333667
10	100	$2^2.5^2$.7.11.13.31.41.211.241.271.2161.9091.	11,41,271,9091
11	100	$2^2.5^2$.67.21649.	
12	100	$2^2.3.5^2$.7.11.13.19.101.9901.52579.333667.	11,333667
2	1000	$2^3.5^3$.73.137	
3	1000	$2^3.3.5^3$.7.13.37.9901	37
4	1000	$2^3.5^3$.17.73.137.	
5	1000	$2^3.5^3$.41.271.3541.9091.27961	41,271,9091
6	1000	$2^3.3.5^3$.7.13.37.73.137.9901.	37
7	1000	$2^3.5^3$.29.239.281.4649.	
8	1000	$2^3.5^3$.17.73.137.353.449.641.1409.69857.	
9	1000	$2^3.3^2.5^3$.7.13.19.37.9901.52579.333667.	37,333667
10	1000	$2^3.3.5^3$.41.73.137.271.3541.9091.27961.	41,271,9091
11	1000	$2^3.5^3$.11.23.89.4093.8779.21649.	11
2	10000	$2^4.5^4$.11.9091	11,9091
3	10000	$2^4.3.5^4$.31.37.	37
4	10000	$2^4.5^4$.11.101.3541.9091.27961	11,9091
5	10000	$2^4.5^4$.21401.25601.	
6	10000	$2^4.3.5^4$.7.11.13.31.37.211.241.2161.9091.	11,37,9091
7	10000	$2^4.5^4$.71.239.4649.123551.	
8	10000	$2^4.5^4$.11.73.101.137.3541.9091.27961.	11,9091
9	10000	$2^4.3.5^4$.31.37.238681.333667.	37,333667
2	100000	$2^5.5^5$.101.9901	
3	100000	$2^5.3.5^5$.19.52579.333667	333667
4	100000	$2^5.5^5$.73.101.137.9901..	
5	100000	$2^5.5^5$.31.41.211.241.271.2161.9091..	41,271,9091
6	100000	$2^5.3.5^5$.19.101.9901.52579.333667..	333667
7	100000	$2^5.5^5$.7.43.127.239.1933.2689.4649..	43
8	100000	$2^5.5^5$.17.73.101.137.9901..	
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THE SMARANDACHE FRIENDLY NATURAL NUMBER PAIRS

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Abstract. In this paper we completely determinate all the Smarandache friendly natural number pairs.

Key words: Smarandache friendly natural number pair, Pell equation, positive integer solution

Let **Z**, **N** be the sets of all integers and positive integers respectively. Let a, b be two positive integers with $a<b$. Then the pair (a, b) is called a Smarandache friendly natural number pair if

(1) $$a+(a+1)+\cdots\cdots+b=ab.$$

For example, (1, 1), (3, 6), (15, 35), (85, 204) are Smarandache friendly natural number pairs. In [2], Murthy showed that there exist infinitely many such pairs. In this paper we shall completely determinate all Smarandache friendly natural number pairs.

Let

(2) $$\alpha=1+\sqrt{2},\ \ \beta=1-\sqrt{2}.$$

For any positive integer n, let

(3) $$P(n)=\frac{1}{2}\left(\alpha^n+\beta^n\right),\ Q(n)=\frac{1}{2\sqrt{2}}\left(\alpha^n-\beta^n\right).$$

Notice that $1+\sqrt{2}$ and $\left(1+\sqrt{2}\right)^2=3+2\sqrt{2}$ are the fundamental solutions of Pell equations

(4) $$x^2-2y^2=-1, x, y\in \mathbf{N},$$

and

(5) $$x^2-2y^2=1, x, y\in \mathbf{N},$$

respectively. By [1, Chapter 8], we obtain the following two lemmas immediately.

Lemma 1. All solutions (x, y) of (4) are given by

(6) $$x=P(2m+1), y= Q(2m+1), \quad m\in \mathbf{Z}, m\geqslant 0.$$

Lemma 2. All solutions (x, y) of (5) are given by

(7) $$x=P(2m), y= Q(2m), m\in \mathbf{N}.$$

We now prove a general result as follows.

Theorem. If (a, b) is a Smarandache friendly natural number pair, then either

(8) $$a=(P(2m+1)+2Q(2m+1))Q(2m+1),$$
$$b=(P(2m+1)+2Q(2m+1))(P(2m+1)+Q(2m+1)), m\in \mathrm{Z}, m\geq 0$$

or

(9) $$a=(P(2m)+Q(2m))P(2m),$$
$$b=(P(2m)+Q(2m))(P(2m)+2Q(2m)), m\in \mathrm{N}.$$

Proof. Let (a, b) be a Smarandache friendly natural number pair. Since

(10) $$a+(a+1)+\ldots+b=(1+2+\ldots+b)-(1+2+\ldots+(a-1))$$
$$=\frac{1}{2}b(b+1)-\frac{1}{2}a(a-1)=\frac{1}{2}(b+a)(b-a+1),$$

we get from (1) that

(11) $(b+a)(b-a+1)=2ab.$

Let $d=\gcd(a, b)$. Then we have

(12) $a=da_1, \quad b=db_1,$

where a_1, b_1 are positive integers satisfying

(13) $a_1<b_1, \gcd(a_1, b_1)=1.$

Substitute (12) into (11), we get

(14) $$(b_1+a_1)(d(b_1-a_1)+1)=2da_1b_1.$$

Since gcd $(a_1, b_1)=1$ by (13), we get gcd $(a_1b_1, a_1+b_1)=1$. Similarly, we have gcd $(d, d(b_1-a_1)+1)=1$. Hence, we get from (14) that

(15) $$d \mid b_1+a_1, \; a_1b_1 \mid d(b_1-a_1)+1.$$

Therefore, by (14) and (15), we obtain either

(16) $b_1+a_1=d, \quad d(b_1-a_1)+1=2a_1b_1$

or

(17) $b_1+a_1=2d, \quad d(b_1-a_1)+1=a_1b_1$

If (16) holds, then we have

(18) $$d(b_1-a_1)+1=(b_1+a_1)(b_1-a_1)+1=b_1^2-a_1^2+1=2a_1b_1.$$

whence we get

(19) $$(b_1-a_1)^2-2a_1^2=-1.$$

It implies that $(x, y)=(b_1-a_1, a_1)$ is a solution of (4). Thus, by Lemma 1, we get (8) by (16).

If (17) holds, then we have

$$d(b_1 - a_1) + 1 = \frac{1}{2}(b_1 + a_1)(b_1 - a_1) + 1 = \frac{1}{2}(b_1^2 - a_1^2) + 1 = a_1 b_1. \tag{20}$$

Since gcd $(a_1, b_1)=1$ by (13), we see from (17) that both a_1 and b_1 are odd. If implies that $(b_1-a_1)/2$ is a positive integer. By (20), we get

$$a_1^2 - 2\left(\frac{b_1 - a_1}{2}\right)^2 = 1. \tag{21}$$

We find from (21) that $(x, y)=(a_1, (b_1-a_1)/2)$ is a solution of (5). Thus, by Lemma 2, we obtain (9) by (17). The theorem is proved.

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ON THE 17-th SMARANDACHE'S PROBLEM

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The 17-th problem from [1] (see also 22-nd problem from [2]) is the following:

17. Smarandache's digital products:

$$\underbrace{0,1,2,3,4,5,6,7,8,9}, \underbrace{0,1,2,3,4,5,6,7,8,9}, \underbrace{0,2,4,6,8,19,12,14,16,18},$$

$$\underbrace{0,3,6,9,12,15,18,21,24,27}, \underbrace{0,4,8,12,16,20,24,28,32,36}, \underbrace{0,5,10,15,20,25...}$$

($d_p(n)$ is the product of digits.)

Let the fixed natural number n have the form $n = \overline{a_1 a_2 ... a_k}$, where $a_1, a_2, ..., a_k \in \{0, 1, ..., 9\}$ and $a_1 \geq 1$. Therefore,

$$n = \sum_{i=1}^{k} a_i 10^{i-1}.$$

Hence, $k = [\log_{10} n] + 1$ and

$$a_1(n) \equiv a_1 = [\frac{n}{10^{k-1}}],$$

$$a_2(n) \equiv a_2 = [\frac{n - a_1 10^{k-1}}{10^{k-2}}],$$

$$a_3(n) \equiv a_3 = [\frac{n - a_1 10^{k-1} - a_2 10^{k-2}}{10^{k-3}}],$$

$$\cdots$$

$$a_{[\log_{10} n]}(n) \equiv a_{k-1} = [\frac{n - a_1 10^{k-1} - ... - a_{k-2} 10^2}{10}],$$

$$a_{[\log_{10} n]+1}(n) \equiv a_k = n - a_1 10^{k-1} - ... - a_{k-1} 10.$$

Obviously, $k, a_1, a_2, ..., a_k$ are functions only of n. Therefore,

$$d_p(n) = \prod_{i=1}^{[\log_{10} n]+1} a_i(n).$$

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ON THE 46-th SMARANDACHE'S PROBLEM

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The 46-th problem from [1] is the following:

Smarandache's prime additive complements:

$$1,0,0,1,0,1,0,3,2,1,0,1,0,3,2,1,0,1,0,3,2,1,0,1,0,5,4,3,2,1,0,1,0,$$

$$5,4,3,2,1,0,3,2,1,0,5,4,3,2,1,0,...$$

(For each n to find the smallest k such that $n+k$ is prime.)

Obviously, the members of the above sequence are differences between first prime number bigger or equal to the current natural number n and the same n. It is well known that the number of primes smaller or equal to n is $\pi(n)$. Therefore, the prime number smaller or equal to n is $p_{\pi(n)}$. Hence, the prime number bigger or equal to n is the next prime number, i.e., $p_{\pi(n)+1}$. Finally, the n-th member of the above sequence will be equal to

$$\begin{cases} p_{\pi(n)+1} - n, & \text{if } n \text{ is not a prime number} \\ 0, & \text{otherwise} \end{cases}$$

We shall note that in [2] the author gives the following new formula p_n for every natural number n:

$$p_n = \sum_{i=0}^{C(n)} sg(n - \pi(i)),$$

where $C(n) = [\frac{n^2+3n+4}{4}]$ (for $C(n)$ see [3]) and

$$sg(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases},$$

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ON THE DIVISOR PRODUCTS AND PROPER DIVISOR PRODUCTS SEQUENCES*

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ABSTRACT. Let n be a positive integer, $p_d(n)$ denotes the product of all positive divisors of n, $q_d(n)$ denotes the product of all proper divisors of n. In this paper, we study the properties of the sequences $\{p_d(n)\}$ and $\{q_d(n)\}$, and prove that the Makowski & Schinzel conjecture hold for the sequences $\{p_d(n)\}$ and $\{q_d(n)\}$.

1. INTRODUCTION

Let n be a positive integer, $p_d(n)$ denotes the product of all positive divisors of n. That is, $p_d(n) = \prod_{d|n} d$. For example, $p_d(1) = 1$, $p_d(2) = 2$, $p_d(3) = 3$, $p_d(4) = 8$, $p_d(5) = 5$, $p_d(6) = 36$, $\cdots$, $p_d(p) = p$, $\cdots$. $q_d(n)$ denotes the product of all proper divisors of n. That is, $q_d(n) = \prod_{d|n,d<n} d$. For example, $q_d(1) = 1$, $q_d(2) = 1$, $q_d(3) = 1$, $q_d(4) = 2$, $q_d(5) = 1$, $q_d(6) = 6$, $\cdots$. In problem 25 and 26 of [1], Professor F.Smarandach asked us to study the properties of the sequences $\{p_d(n)\}$ and $\{q_d(n)\}$. About this problem, it seems that none had studied it, at least we have not seen such a paper before. In this paper, we use the elementary methods to study the properties of the sequences $\{p_d(n)\}$ and $\{q_d(n)\}$, and prove that the Makowski & Schinzel conjecture hold for $p_d(n)$ and $q_d(n)$. That is, we shall prove the following:

Theorem 1. *For any positive integer n, we have the inequality*

$$\sigma(\phi(p_d(n))) \geq \frac{1}{2} p_d(n),$$

where $\phi(k)$ is the Euler's function and $\sigma(k)$ is the divisor sum function.

Theorem 2. *For any positive integer n, we have the inequality*

$$\sigma(\phi(q_d(n))) \geq \frac{1}{2} q_d(n).$$

Key words and phrases. Makowski & Schinzel conjecture; Divisor and proper divisor product.
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2. Some Lemmas

To complete the proof of the Theorems, we need the following two Lemmas:

Lemma 1. *For any positive integer n, we have the identities*

$$p_d(n) = n^{\frac{d(n)}{2}} \quad \text{and} \quad q_d(n) = n^{\frac{d(n)}{2}-1},$$

where $d(n) = \sum_{d|n} 1$ *is the divisor function.*

Proof. From the definition of $p_d(n)$ we know that

$$p_d(n) = \prod_{d|n} d = \prod_{d|n} \frac{n}{d}.$$

So by this formula we have

$$p_d^2(n) = \prod_{d|n} n = n^{d(n)}. \tag{1}$$

From (1) we immediately get

$$p_d(n) = n^{\frac{d(n)}{2}}$$

and

$$q_d(n) = \prod_{d|n, d<n} d = \frac{\prod_{d|n} d}{n} = n^{\frac{d(n)}{2}-1}.$$

This completes the proof of Lemma 1.

Lemma 2. *For any positive integer n, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ with $\alpha_i \geq 2$ $(i = 1, 2, \cdots, s)$, $p_j (j = 1, 2, \cdots, s)$ are some different primes with $p_1 < p_2 < \cdots < p_s$, then we have the estimate*

$$\sigma(\phi(n)) \geq \frac{6}{\pi^2} n.$$

Proof. From the properties of the Euler's function we have

$$\begin{aligned} \phi(n) &= \phi(p_1^{\alpha_1})\phi(p_2^{\alpha_2}) \cdots \phi(p_s^{\alpha_s}) \\ &= p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \cdots p_s^{\alpha_s - 1} (p_1 - 1)(p_2 - 1) \cdots (p_s - 1). \end{aligned} \tag{2}$$

Let $(p_1 - 1)(p_2 - 1) \cdots (p_s - 1) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_s^{\beta_s} q_1^{r_1} q_2^{r_2} \cdots q_t^{r_t}$, where $\beta_i \geq 0$, $i = 1, 2, \cdots, s$, $r_j \geq 1$, $j = 1, 2, \cdots, t$ and $q_1 < q_2 < \cdots < q_t$ are different primes. Then

from (2) we have

$$\begin{aligned}
\sigma(\phi(n)) &= \sigma(p_1^{\alpha_1+\beta_1-1}p_2^{\alpha_2+\beta_2-1}\cdots p_s^{\alpha_s+\beta_s-1}q_1^{r_1}q_2^{r_2}\cdots q_t^{r_t}) \\
&= \prod_{i=1}^{s}\frac{p_i^{\alpha_i+\beta_i}-1}{p_i-1}\prod_{j=1}^{t}\frac{q_j^{r_j+1}-1}{q_j-1} \\
&= p_1^{\alpha_1+\beta_1}p_2^{\alpha_2+\beta_2}\cdots p_s^{\alpha_s+\beta_s}q_1^{r_1}q_2^{r_2}\cdots q_t^{r_t}\prod_{i=1}^{s}\frac{1-\frac{1}{p_i^{\alpha_i+\beta_i}}}{p_i-1}\prod_{j=1}^{t}\frac{1-\frac{1}{q_j^{r_j+1}}}{1-\frac{1}{q_j}} \\
&= n\prod_{i=1}^{s}\left(1-\frac{1}{p_i^{\alpha_i+\beta_i}}\right)\prod_{j=1}^{t}\frac{1-\frac{1}{q_j^{r_j+1}}}{1-\frac{1}{q_j}} \\
&= n\prod_{i=1}^{s}\left(1-\frac{1}{p_i^{\alpha_i+\beta_i}}\right)\prod_{j=1}^{t}\left(1+\frac{1}{q_j}+\cdots+\frac{1}{q_j^{r_j}}\right) \\
&\geq n\prod_{i=1}^{s}\left(1-\frac{1}{p_i^{\alpha_i+\beta_i}}\right) \\
&\geq n\prod_{i=1}^{s}\left(1-\frac{1}{p_i^2}\right) \\
&\geq n\prod_{p}\left(1-\frac{1}{p^2}\right).
\end{aligned}$$

Noticing $\prod_p \frac{1}{1-\frac{1}{p^2}} = \sum_{n=1}^{\infty}\frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6}$, we immediately get

$$\sigma(\phi(n)) \geq n\cdot\prod_{p}\left(1-\frac{1}{p^2}\right) = \frac{6}{\pi^2}n.$$

This completes the proof of Lemma 2.

3. Proof of the Theorems

In this section, we shall complete the proof of the Theorems. First we prove Theorem 1. We separate n into prime and composite number two cases. If n is a prime, then $d(n) = 2$. This time by Lemma 1 we have

$$p_d(n) = n^{\frac{d(n)}{2}} = n.$$

Hence, from this formula and $\phi(n) = n-1$ we immediately get

$$\sigma(\phi(p_d(n))) = \sigma(n-1) = \sum_{d|n-1} d \geq n-1 \geq \frac{n}{2} = \frac{1}{2}p_d(n).$$

If n is a composite number, then $d(n) \geq 3$. If $d(n) = 3$, we have $n = p^2$, where p is a prime. So that

$$p_d(n) = n^{\frac{d(n)}{2}} = p^{d(n)} = p^3. \tag{3}$$

From Lemma 2 and (3) we can easily get the inequality

$$\sigma\left(\phi\left(p_d(n)\right)\right) = \sigma\left(\phi\left(p^3\right)\right) \geq \frac{6}{\pi^2}p^3 \geq \frac{1}{2}p_d(n).$$

If $d(n) \geq 4$, let $p_d(n) = n^{\frac{d(n)}{2}} = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ with $p_1 < p_2 < \cdots < p_s$, then we have $\alpha_i \geq 2, i = 1, 2, \cdots, s$. So from Lemma 2 we immediately obtain the inequality

$$\sigma\left(\phi\left(p_d(n)\right)\right) \geq \frac{6}{\pi^2}p_d(n) \geq \frac{1}{2}p_d(n).$$

This completes the proof of Theorem 1.

The proof of Theorem 2. We also separate n into two cases. If n is a prime, then we have

$$q_d(n) = n^{\frac{d(n)}{2}-1} = 1.$$

From this formula we have

$$\sigma\left(\phi\left(q_d(n)\right)\right) = 1 \geq \frac{1}{2}q_d(n).$$

If n is a composite number, we have $d(n) \geq 3$, then we discuss the following four cases. First, if $d(n) = 3$, then $n = p^2$, where p is a prime. So we have

$$q_d(n) = n^{\frac{d(n)}{2}-1} = p^{d(n)-2} = p.$$

From this formula and the proof of Theorem 1 we easily get

$$\sigma\left(\phi\left(q_d(n)\right)\right) \geq \frac{1}{2}q_d(n).$$

Second, if $d(n) = 4$, from Lemma 1 we may get

$$q_d(n) = n^{\frac{d(n)}{2}-1} = n \tag{4}$$

and $n = p^3$ or $n = p_1p_2$, where p, p_1 and p_2 are primes with $p_1 < p_2$. If $n = p^3$, from (4) and Lemma 2 we have

$$\begin{aligned}\sigma\left(\phi\left(q_d(n)\right)\right) &= \sigma\left(\phi(n)\right) = \sigma\left(\phi(p^3)\right) \\ &\geq \frac{1}{2}p^3 = \frac{1}{2}q_d(n).\end{aligned} \tag{5}$$

If $n = p_1p_2$, we consider $p_1 = 2$ and $p_1 > 2$ two cases. If $2 = p_1 < p_2$, then $p_2 - 1$ is an even number. Supposing $p_2 - 1 = p_1^{\beta_1}p_2^{\beta_2}q_1^{r_1}\cdots q_t^{r_t}$ with $q_1 < q_2 < \cdots < q_t$,

$q_i (i = 1, 2, \cdots, t)$ are different primes and $r_j \geq 1$ $(j = 1, 2, \cdots, t)$, $\beta_1 \geq 1$, $\beta_2 \geq 0$. Note that the proof of Lemma 2 and (4) we can obtain

$$\begin{aligned}
\sigma(\phi(q_d(n))) &= \sigma(\phi(n)) \\
&= n \prod_{i=1}^{2} \left(1 - \frac{1}{p_i^{1+\beta_i}}\right) \prod_{j=1}^{t} \left(1 + \frac{1}{q_j} + \cdots + \frac{1}{q_j^{r_j}}\right) \\
&\geq n \left(1 - \frac{1}{p_1^2}\right)\left(1 - \frac{1}{p_2}\right) \\
&\geq n(1 - \frac{1}{4})(1 - \frac{1}{3}) \\
&= \frac{1}{2} q_d(n). \qquad (6)
\end{aligned}$$

If $2 < p_1 < p_2$, then both $p_1 - 1$ and $p_2 - 1$ are even numbers. Let $(p_1 - 1)(p_2 - 1) = p_1^{\beta_1} p_2^{\beta_2} q_1^{r_1} q_2^{r_2} \cdots q_t^{r_t}$ with $q_1 < q_2 < \cdots < q_t, q_i (i = 1, 2, \cdots, t)$ are different primes and $r_j \geq 1 (j = 1, 2, \cdots, t), \beta_1, \beta_2 \geq 0$, then we have $q_1 = 2$ and $r_1 \geq 2$. So from the proof of Lemma 2 and (4) we have

$$\begin{aligned}
\sigma(\phi(q_d(n))) &= \sigma(\phi(n)) \\
&= n \prod_{i=1}^{2} \left(1 - \frac{1}{p_i^{1+\beta_i}}\right) \prod_{j=1}^{t} \left(1 + \frac{1}{q_j} + \cdots + \frac{1}{q_j^{r_j}}\right) \\
&\geq n \prod_{i=1}^{2} \left(1 - \frac{1}{p_i}\right)\left(1 + \frac{1}{2} + \frac{1}{2^2}\right) \\
&\geq n \prod_{i=1}^{2} \left(1 - \frac{1}{p_i}\right)(1 + \frac{1}{3})(1 + \frac{1}{5}) \\
&\geq n \prod_{i=1}^{2} \left[(1 - \frac{1}{p_i})(1 + \frac{1}{p_i})\right] \\
&\geq n \prod_{p} \left(1 - \frac{1}{p^2}\right) \\
&\geq n \frac{6}{\pi^2} \\
&\geq \frac{1}{2} q_d(n). \qquad (7)
\end{aligned}$$

Combining (5), (6) and (7) we obtain

$$\sigma(\phi(q_d(n))) \geq \frac{1}{2} q_d(n) \quad \text{if} \quad d(n) = 4.$$

Third, if $d(n) = 5$, we have $n = p^4$, where p is a prime. Then from Lemma 1 and Lemma 2 we immediately get

$$\sigma(\phi(q_d(n))) = \sigma(\phi(p^6)) \geq \frac{6}{\pi^2} p^6 = \frac{1}{2} q_d(n).$$

Finaly, if $d(n) \geq 6$, then from Lemma 1 and Lemma 2 we can easily obtain

$$\sigma(\phi(q_d(n))) \geq \frac{1}{2} q_d(n).$$

This completes the proof of Theorem 2.

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SOME SMARANDACHE-TYPE MULTIPLICATIVE FUNCTIONS

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This note considers eleven particular families of interrelated multiplicative functions, many of which are listed in Smarandache's problems.

These are multiplicative in the sense that a function f(n) has the property that for any two coprime positive integers a and b, i.e. with a highest common factor (also known as greatest common divisor) of 1, then f(a*b)=f(a)*f(b). It immediately follows that f(1)=1 unless all other values of f(n) are 0. An example is d(n), the number of divisors of n. This multiplicative property allows such functions to be uniquely defined on the positive integers by describing the values for positive integer powers of primes. $d(p^i)=i+1$ if $i>0$; so $d(60) = d(2^2*3^1*5^1) = (2+1)*(1+1)*(1+1) = 12$.

Unlike d(n), the sequences described below are a restricted subset of all multiplicative functions. In all of the cases considered here, $f(p^i)=p^{g(i)}$ for some function g which does not depend on p.

	Definition	**Multiplicative with p^i->p^...**
$A_m(n)$	Number of solutions to $x^m == 0 \pmod n$	i-ceiling[i/m]
$B_m(n)$	Largest m^{th} power dividing n	m*floor[i/m]
$C_m(n)$	m^{th} root of largest mth power dividing n	floor[i/m]
$D_m(n)$	m^{th} power-free part of n	i-m*floor[i/m]
$E_m(n)$	Smallest number x (x>0) such that n*x is a perfect m^{th} power (Smarandache m^{th} power complements)	m*ceiling[i/m]-i
$F_m(n)$	Smallest m^{th} power divisible by n divided by largest m^{th} power which divides n	m*(ceiling[i/m]-floor[i/m])
$G_m(n)$	m^{th} root of smallest m^{th} power divisible by n divided by largest m^{th} power which divides n	ceiling[i/m]-floor[i/m]
$H_m(n)$	Smallest m^{th} power divisible by n (Smarandache ^m function (numbers))	m*ceiling[i/m]
$J_m(n)$	m^{th} root of smallest mth power divisible by n (Smarandache Ceil Function of m^{th} Order)	ceiling[i/m]
$K_m(n)$	Largest m^{th} power-free number dividing n	min(i,m-1)

	(Smarandache m^{th} power residues)	
$L_m(n)$	n divided by largest m^{th} power-free number dividing n	max(0,i-m+1)

Relationships between the functions

Some of these definitions may appear to be similar; for example $D_m(n)$ and $K_m(n)$, but for m>2 all of the functions are distinct (for some values of n given m). The following relationships follow immediately from the definitions:

$$B_m(n)=C_m(n)^m \quad (1)$$
$$n=B_m(n)*D_m(n) \quad (2)$$
$$F_m(n)=D_m(n)*E_m(n) \quad (3)$$
$$F_m(n)=G_m(n)^m \quad (4)$$
$$H_m(n)=n*E_m(n) \quad (5)$$
$$H_m(n)=B_m(n)*F_m(n) \quad (6)$$
$$H_m(n)=J_m(n)^m \quad (7)$$
$$n=K_m(n)*L_m(n) \quad (8)$$

These can also be combined to form new relationships; for example from (1), (4) and (7) we have

$$J_m(n)=C_m(n)*G_m(n) \quad (9)$$

Further relationships can also be derived easily. For example, looking at $A_m(n)$, a number x has the property $x^m \equiv 0 \pmod n$ if and only if x^m is divisible by n or in other words a multiple of $H_m(n)$, i.e. x is a multiple of $J_m(n)$. But $J_m(n)$ divides into n, so we have the pretty result that:

$$n=J_m(n)*A_m(n) \quad (10)$$

We could go on to construct a wide variety of further relationships, such as using (5), (7) and (10) to produce:

$$n^{m-1}=E_m(n)*A_m(n)^m \quad (11)$$

but instead we will note that $C_m(n)$ and $J_m(n)$ are sufficient to produce all of the functions from $A_m(n)$ through to $J_m(n)$:

$$A_m(n)=n/J_m(n) \quad (12)$$
$$B_m(n)=C_m(n)^m$$

$$C_m(n)=C_m(n)$$

$$D_m(n)=n/C_m(n)^m \quad (13)$$

$$E_m(n)=J_m(n)^m/n \quad (14)$$

$$F_m(n)=(J_m(n)/C_m(n))^m \quad (15)$$

$$G_m(n)=J_m(n)/C_m(n) \quad (16)$$

$$H_m(n)=J_m(n)^m$$

$$J_m(n)=J_m(n)$$

Clearly we could have done something similar by choosing one element each from two of the sets {A,E,H,J}, {B,C,D}, and {F,G}. The choice of C and J is partly based on the following attractive property which further deepens the multiplicative nature of these functions.

If m=a*b then:

$$C_m(n)=C_a(C_b(n)) \quad (17)$$

$$J_m(n)=J_a(J_b(n)) \quad (18)$$

Duplicate functions when m=2 ...

When m=2, $D_2(n)$ is square-free and $F_2(n)$ is the smallest square which is a multiple of $D_2(n)$, so

$$F_2(n)=D_2(n)^2 \quad (19)$$

Using (3) and (4) we then have:

$$D_2(n)=E_2(n)=G_2(n) \quad (20)$$

and from (13) and (14) we have

$$n=C_2(n)*J_2(n) \quad (21)$$

so from (10) we get

$$A_2(n)=C_2(n) \quad (22)$$

... and when m=1

If m=1, all the functions described either produce 1 or n. The analogue of (20) is still true with

$$D_1(n)=E_1(n)=G_1(n)=1 \quad (23)$$

but curiously the analogue of (22) is not, since:

$$A_1(n)=1 \quad (24)$$

$$C_1(n)=n \quad (25)$$

The two remaining functions

All this leaves two slightly different functions to be considered: $K_m(n)$ and $L_m(n)$. They have little connection with the other sequences except for the fact that since $G_m(n)$ is square-free, and divides $D_m(n)$, $E_m(n)$, $F_m(n)$, and $G_m(n)$, none of which have any factor which is a higher power than m, we get:

$$G_m(n)=J_m(D_m(n))=J_m(E_m(n))=J_m(F_m(n))=J_m(G_m(n))=K_2(D_m(n))=K_2(E_m(n))=K_2(F_m(n))=K_2(G_m(n)) \quad (26)$$

and so with (8) and (10)

$$n/G_m(n)=A_m(D_m(n))=A_m(E_m(n))=A_m(F_m(n))=A_m(G_m(n))=L_2(D_m(n))=L_2(E_m(n))=L_2(F_m(n))=L_2(G_m(n)) \quad (27)$$

We also have the related convergence property that for any y, there is a z (e.g. floor[$\log_2(n)$]) for which

$$G_m(n)=J_m(n)=K_2(n) \text{ for any } n<=y \text{ and any } m>z \quad (28)$$

$$A_m(n)=L_2(n) \text{ for any } n<=y \text{ and any } m>z \quad (29)$$

There is a simple relation where

$$L_m(n)=L_a(L_b(n)) \text{ if } m+1=a+b \text{ and } a,b>0 \quad (29)$$

and in particular

$$L_m(n)=L_{m-1}(L_2(n)) \text{ if } m>1 \quad (30)$$

$$L_3(n)=L_2(L_2(n)) \quad (31)$$

$$L_4(n)=L_2(L_2(L_2(n))) \quad (32)$$

so with (8) we also have

$$K_m(n)=K_b(n)*K_a(n/K_b(n)) \text{ if } m+1=a+b \text{ and } a,b>0 \quad (33)$$

$$K_m(n)=K_{m-1}(n)*K_2(n/K_{m-1}(n)) \text{ if } m>1 \quad (34)$$

$$K_3(n)=K_2(n)*K_2(n/K_2(n)) \quad (35)$$

$$K_4(n)=K_2(n)*K_2(n/K_2(n))*K_2(n/(K_2(n)*K_2(n/K_2(n)))) \quad (36)$$

Recording the functions

The values of all these functions for n up from n=1 to about 70 or more are listed in Neil Sloane's Online Encyclopedia of Integer Sequences for m=2, 3 and 4:

	m=1	m=2	m=3	m=4	m>=x and n<2^x
$A_m(n)$	1	A000188	A000189	A000190	$L_2(n)$
$B_m(n)$	n	A008833	A008834	A008835	1
$C_m(n)$	n	A000188	A053150	A053164	1
$D_m(n)$	1	A007913	A050985	A053165	n
$E_m(n)$	1	A007913	A048798	A056555	$K_2(n)^m/n$
$F_m(n)$	1	A055491	A056551	A056553	$K_2(n)^m$
$G_m(n)$	1	A007913	A056552	A056554	$K_2(n)$
$H_m(n)$	n	A053143	A053149	A053167	$K_2(n)^m$
$J_m(n)$	n	A019554	A019555	A053166	$K_2(n)$
$K_m(n)$	1	A007947	A007948	A058035	n
$L_m(n)$	n	A003557	A062378	A062379	1

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THE PSEUDO-SMARANDACHE FUNCTION

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Abstract:

The Pseudo-Smarandache Function is part of number theory. The function comes from the Smarandache Function. The Pseudo-Smarandache Function is represented by Z(n) where n represents any natural number. The value for a given Z(n) is the smallest integer such that 1+2+3+ . . . + Z(n) is divisible by n. Within the Pseudo-Smarandache Function, there are several formulas which make it easier to find the Z(n) values.

Formulas have been developed for most numbers including:

a) p, where p equals a prime number greater than two;
b) b, where p equals a prime number, x equals a natural number, and $b=p^x$;
c) x, where x equals a natural number, if x/2 equals an odd number greater than two;
d) x, where x equals a natural number, if x/3 equals a prime number greater than three.

Therefore, formulas exist in the Pseudo-Smarandache Function for all values of b except for the following:

a) x, where x = a natural number, if x/3 = a nonprime number whose factorization is not 3^x;
b) multiples of four that are not powers of two.

All of these formulas are proven, and their use greatly reduces the effort needed to find Z(n) values.

Keywords:
Smarandache Function, Pseudo-Smarandache Function, Number Theory, Z(n), g(d), g[Z(n)].

Introduction.

The Smarandache (sma-ran-da-ke) Functions, Sequences, Numbers, Series, Constants, Factors, Continued Fractions, Infinite Products are a branch of number theory. There are very interesting patterns within these functions, many worth studying sequences. The name "Pseudo-Smarandache Function" comes from the Smarandache function. [2] The Smarandache Function was named after a Romanian mathematician and poet, Florentin Smarandache. [1] The Smarandache Function is represented as S(n) where n is any natural number. S(n) is defined as the smallest m, where m represents any natural number, such that m! is divisible by n.
To be put simply, the Smarandache Function differs from the Pseudo-Smarandache Function in that in the Smarandache Function, multiplication is used in the form of factorials; in the Pseudo -Smarandache Function, addition is used in the place of the multiplication. The Pseudo-Smarandache Function is represented by Z(n) where n represents all natural numbers. The value for a given Z(n) is the smallest integer such that 1+2+3+ . . . + Z(n) is divisible by n.

d	g(d)
1	1
2	3
3	6
4	10
5	15
6	21
7	28
8	36
9	45
10	55

Figure 1: The first ten g(d) values.

Background

As previously stated, the value for a given Z(n) is the smallest integer such that 1+2+3+ . . . + Z(n) is divisible by n. Because consecutive numbers are being added, the sum of 1+2+3+ . . . + Z(n) is a triangle number. Triangle numbers are numbers that can be written in the form [d(d+1)]/2 where d equals any natural number. When written in this form, two consecutive numbers must be present in the numerator. In order to better explain the Z(n) function, the g(d) function has been introduced where g(d)=⌈d(d+1)⌉/2.

n	Z(n)	g[Z(n)]
1	1	1
2	3	6
3	2	3
4	7	28
5	4	10
6	3	6
7	6	21
8	15	120
9	8	36
10	4	10
11	10	55
12	8	36
13	12	78
14	7	28
15	5	15
16	31	496
17	16	136
18	8	36
19	18	171
20	15	120

Figure 2: The first 20 Z(n) and g[Z(n)] values.

g[Z(n)] values are defined as g(d) values where d equals Z(n). Because of this, it is important to note that all g[Z(n)] values are g(d) values but special ones because they correspond to a particular n value. Since g(d)=[d(d+1)]/2, g[Z(n)]=[Z(n)[Z(n)+1]/2]. Because g(d) is evenly divisible by n, and all g[Z(n)] are also g(d) values, g[Z(n)] is evenly divisible by n. Therefore, the expression [Z(n)[Z(n)+1]/2] can be shortened to n*k (where k is any natural number). If k=x/2 (where x is any natural number) then [Z(n)[Z(n)+1]/2]=(n*x)/2, and the "general form" for a g[Z(n)] value is (n*x)/2. Again, since (n*x)/2 represents a g(d) value, it must contain all of the characteristics of g(d) values. As said before, all g(d) values, when written in the form [d(d+1)]/2, must be able to have two consecutive numbers in their numerator. Therefore, in the expression (n*x)/2, n and x must be consecutive, or they must be able to be factored and rearranged to yield two consecutive numbers. For some values of n, g[Z(n)]=(n*x)/2 where x is much less than n (and they aren't consecutive). This is possible because for certain number combinations n and x can be factored and rearranged in a way that makes them consecutive. For example, Z(n=12) is 8, and g[Z(12)] is 36. This works because the original equation was (12*6)/2=36, but after factoring and rearranging 12 and 6, the equation can be rewritten as (8*9)/2=36.

The Pseudo-Smarandache Function specifies that only positive numbers are used. However, what if both d and n were less than zero? g(d) would then represent the sum of the numbers from d to –1. Under these circumstances, Z(n) values are the same as the Z(n) values in the "regular" system (where all numbers are greater than one) except they are negated. This means that Z(-n)=-[Z(n)]. This occurs because between the positive system and the negative system, the g(d) values are also the same, just negated. For example, g(4)=4+3+2+1=10 and g(-4)= -4+ -3+ -2+ -1=-10. Therefore, the first g(d) value which is evenly divisible by a given value of n won't change between the positive system and the negative system.

Theorem 1

If 'p' is a prime number greater than two, then Z(p)=p-1

Example:

p	Z(p)
3	2
5	4
7	6
11	10
13	12
17	16
19	18
23	22
27	26
29	28

Figure 3: The first 10 Z(p) values.

Proof:

Since we are dealing with specific p values, rather than saying g[Z(n)]=(n*x)/2, we can now say g[j(p)]=(p*x)/2. Therefore, all that must be found is the lowest value of x that is consecutive to p, or the lowest value of x that can be factored and rearranged to be consecutive to p. Since p is prime, it has no natural factors other than one and itself. Therefore, the lowest value of x that is consecutive to p is p-1. Therefore Z(p)=p-1.

Theorem 2

If x equals any natural number, p equals a prime number greater than two, and b equals p^x, then Z(b)=b-1

Example:

b	Z(b)
3	2
9	8
27	26
81	80
243	242
729	728

b	Z(b)
5	4
25	24
125	124
625	624
3125	3124
15625	15624

b	Z(b)
7	6
49	48
343	342
2401	2400
16807	16806
117649	117648

Figure 4: the first Z(b) values for different primes.

Proof:

The proof for this theorem is similar to the proof of theorem 2. Again, the g(d) function is made up of the product of two consecutive numbers divided by two. Since b's roots are the same, it is impossible for something other than one less than b itself to produce to consecutive natural numbers (even when factored and rearranged). For example, g[Z(25)]=(25*x)/2. When trying to find numbers less than 24 which can be rearranged to make two consecutive natural numbers this becomes g[Z(25)]=(5*5*x)/2. There is no possible value of x (that is less than 24) that can be factored and multiplied into 5*5 to make two consecutive natural numbers. This is because 5 and 5 are prime and equal. They can't be factored as is because the have no divisors. Also, there is no value of x that can be multiplied and rearranged into 5*5, again, because they are prime and equal.

Theorem 3

If x equals two to any natural power, then Z(x)=2x-1.

Example:

X	Z(x)
2	3
4	7
8	15
16	31
32	63
64	127
128	255
256	511
512	1023
1024	2047
2048	4095
4096	8191
8192	16383
16384	32767
32768	65535

Figure 5: The first six Z(x) values.

Proof:

According to past logic, it may seem like Z(x) would equal x-1. However, the logic changes when dealing with even numbers. The reason Z(x)≠x-1 is because (x-1)/2 can not be an integral value because x-1 is odd (any odd number divided by two yields a number with a decimal). Therefore, [x(x-1)]/2 is not an even multiple of x. In order to solve this problem, the numerator has to be multiplied by two. In a sense, an extra two is multiplied into the equation so that when the whole equation is divided by two, the two that was multiplied in is the two that is divided out. That way, it won't effect the "important" part of the equation, the numerator, containing the factor of x. Therefore, the new equation becomes 2[x(x-1)]/2, or [2x(x-1)]/2. The only numbers consecutive to 2x are 2x-1 and 2x+1. Therefore, the smallest two consecutive numbers are 2x-1 and 2x.
Therefore, Z(x)=2x-1.

Theorem 4

If 'j' is any natural number where j/2 equals an odd number greater than two then

$$Z(j) = \begin{cases} \frac{j}{2}-1,\ \frac{j}{2}-1 \textit{ is evenly divisble by } 4 \\ \frac{j}{2},\ \frac{j}{2}-1 \textit{ is not evenly divisble by } 4 \end{cases}$$

Example:

j	Z(j)	j/2	(j/2)-1
6	3	3	2
10	4	5	4
14	7	7	6
18	8	9	8
22	11	11	10
26	12	13	12
30	15	15	14
34	16	17	16
38	19	19	18
42	20	21	20
46	23	23	22
50	24	25	24
54	27	27	26
58	28	29	28
62	31	31	30
66	32	33	32

Figure 6: The first twenty j(z) values.

Proof:

When finding the smallest two consecutive numbers that can be made from a j value, start by writing the general form but instead of writing n substitute j in its place. That means g[Z(j)]=(j*x)/2. The next step is to factor j as far as possible making it easier to see what x must be. This means that g[Z(j)]=(2*j/2*x)/2. Since the equation is divided by two, if left alone as g[Z(j)]=(2*j/2*x)/2, the boldface 2 would get divided out. This falsely indicates that j/2*x (what is remaining after the boldface 2 is divided out) is evenly divisible by j for every natural number value of x. However, j/2*x isn't always evenly divisible by j for every natural number value of x. The two that was just divided out must be kept in the equation so that one of the factors of the g(d) value being made is j. In order to fix this the whole equation must be multiplied by two so that every value of x is evenly divisible by j. In a sense, an extra two is multiplied into the equation so that so that when the whole equation is divided by two, the two that was multiplied in is the two that gets divided out. That way, it won't effect the "important" part of the equation containing the factor of two. Therefore it becomes g[Z(j)]=(2*2*j/2*f)/2 where f represents any natural number. This is done so that even when divided by two there is still one factor of j. At this point, it looks as though the lowest consecutive integers that can be made from g[Z(j)]=(2*2*j/2*f) are (j/2) and(j/2)-1. However, this is only sometimes the case. This is where the formula changes for every other value of j. If (j/2)-1 is evenly divisible by the '2*2' (4), then Z(j)=(j/2)-1. However, if (j/2)-1 is not evenly divisible by 4, then the next lowest integer consecutive to j/2 is (j/2)+1. (Note: If (j/2)-1 is not evenly divisible by 4, then the next lowest integer consecutive to j/2 is (j/2)+1. (Note: If (j/2)-1 is not evenly divisible by four, then (j/2)+1 must be evenly divisible by 4 because 4 is evenly divisible by every other multiple of two.) Therefore, if (j/2)-1 is not evenly divisible by 4 then g[Z(j)]=[(j/2)[(j/2)+1]]/2 or Z(j)=j/2.

Theorem 5

If 'p' is any natural number where p/3 equals a prime number greater than 3 then

$$Z(n) = \begin{cases} \frac{p}{3}-1,\ \frac{p}{3}-1 \textit{ is evenly divisible by } 3 \end{cases}$$

Example:

p	Z(p)	p/3	(p/3)-1
15	5	5	4
21	6	7	6
33	11	11	10
39	12	13	12
51	17	17	16
57	18	19	18
69	23	23	22
87	28	29	28
93	31	31	30
111	36	37	36

Figure 7: The first ten Z(p) values.

Proof:

The proof for this theorem is very similar to the proof for theorem 4. Since p values are being dealt with, p must be substituted into the general form. Therefore, g[Z(p)]=(p*x)/2. Since what made p is already known, p can be factored further so that g[Z(p)]=(3*p/3*x)/2. At this point it looks like the consecutive numbers that will be made out of (the numerator) 3*p/3*x are p/3 and (p/3)-1 (this is because the greatest value already in the numerator is p/3). However, this is only sometimes the case. When p/3-1 is divisible by 3, the consecutive integers in the numerator are p/3 and (p/3)-1. This means that Z(p)=p/3-1 if p/3-1 is evenly divisible by 3. However, if p/3-1 is not divisible by three, the next smallest number that is consecutive to p/3 is (p/3)+1. If (p/3)-1 is not divisible by 3 then (p/3)+1 must be divisible by 3 (see *1 for proof of this statement). Therefore, the consecutive numbers in the numerator are p/3 and (p/3)+1. This means that Z(p) = p/3 if (p/3)-1 is not evenly divisible by three.

Note: Although there is a similar formula for some multiples of the first two primes, this formula does not exist for the next prime number, 5.

3
4
5
6
7
8
9
10
11
12
13

*1 – "If (p/3)-1 is not divisible by 3, then (p/3)+1 must be divisible by 3."

In the table to the left, the underlined values are those that are divisible by three. The bold numbers are those that are divisible by two (even). Since p/3 is prime it cannot be divisible by three. Therefore, the p/3 values must fall somewhere between the underlined numbers. This leaves numbers like 4, 5, 7, 8, 10, 11, etc. Out of these numbers, the only numbers where the number before (or (p/3)-1) is not divisible by three are the numbers that precede the multiples of three. This means that the p/3 values must be the numbers like 5, 8, 11, etc. Since all of these p/3 values precede multiples of 3, (p/3)+1 must be divisible by 3 if (p/3)-1 is not divisible by 3.

Figure 10

Theorem 6

If 'n' equals any natural number, Z(n)≠n.

Proof:

Theorem 6: Part A

If r is any natural odd number, $Z(r) \not\leq 1$

Proof:

When r is substituted into the general form, g[Z(r)=[r*(r-1)]/2. Since r is odd r-1 is even. Therefore, when r-1 is divided by two, an integral value is produced. Therefore, (r*r-1)/2 is an even multiple of r and it is also a g(d) value. Because of this, $Z(r) \not\leq 1$. Since $Z(r) \not\leq 1$, $Z(r) \neq r$.

Theorem 6: Part B

If v is an natural even number, $Z(v) \neq v$.

Proof:

If Z(v) = v, the general form would appear as the following: g[Z(v)]=[v(v+1)]/2. This is not possible because if v is even then v+1 is odd. When v+1 is divided by two, a non-integral value is produced. Therefore, (v*v+1)/2 is not an integral multiple of v. Therefore, $Z(v) \neq v$.

Theorem 7

If w is any natural number except for numbers whose prime factorization equals 2 to any power, $Z(w) < w$.

Proof

As in several other proofs, this proof can be broken down into two separate parts, a part for r values (r is any natural odd number) and one for v values (v is any natural even number). As proven in Theorem 6: Part A, $Z(r) \not\leq 1$. This proves that Z(r) is less than r.

For v values, v must be substituted into the general form in order to be able to see patterns. Therefore, g[Z(v)]=(v*x)/2. Since v is even it must be divisible by two. Therefore, v can be factored making g[Z(v)]=[2*(v/2)*x]/2. Since the numerator is being divided by two, when done with the division, one whole factor of v will not always be left. Therefore, an extra two must be multiplied into the equation so that even when divided by two, there is still one whole factor of v left. Therefore, $g[Z(v)] \not\leq 4*(v/2)*x]/2$. At this point, the equation can be simplified to $g[Z(v)] \not\leq x$. Therefore, x=v-1, and $Z(v) < v-1$. Z(v) is less than v-1 rather than less than or equal to v-1 because as proven in theorem 4, $Z(v) \neq v-1$.

Conclusion

n	Z(n)
12	8
20	15
24	15
28	7
36	8
40	15
44	32
48	32
52	39
56	48

Figure 8

n/3	n	Z(n)
9	27	8
15	45	9
21	63	27
25	75	24
33	99	44
35	105	14
45	135	54
49	147	48
55	165	44
65	195	39

Figure 9

Through researching the relationships between different groups of natural numbers, patterns and formulas have been developed to find Z(n) values for most numbers. Formulas have been developed for most numbers including:

a) p, where p equals a prime number greater than two
b) b, where p equals a prime number, x equals a natural number, and $b=p^x$
c) x, where x equals a natural number, if x/2 equals an odd number greater than two
d) x, where x equals a natural number, if x/3 equals a prime number greater than three

In fact there are only two remaining groups of numbers for which there are no formulas or shortcuts. Formulas exist in the Pseudo-Smarandache Function for all values of b except for the following:

a) multiples of four that that are not powers of two (figure 8)
b) x, where x = a natural number, if x/3 = a nonprime number whose factorization is not 3^x (figure9)

If p equals a prime number greater than two then Z(p)=p-1. If p equals a prime number greater than two, x equals a natural number, and $b=p^x$ then Z(b)=b-1. However, if p=2 then Z(b)=2b-1. If x equals a natural number, and x/2 equals an odd number greater than two then if (x/2)-1 is evenly divisible by four then Z(x)=(x/2)-1. Otherwise, if x/2-1 is not evenly divisible by four then Z(x)=z/2. If x equals a natural number, and x/3 equals a prime number greater than three then if (x/3)-1 is evenly divisible by three then Z(x)=(x/3)-1. Otherwise, if x/3-1 is not evenly divisible by three then Z(x)=x/3. All of these formulas are proven, and their use greatly reduces the effort needed to find Z(n) values.

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ON THE SYMMETRIC SEQUENCE AND ITS SOME PROPERTIES*

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ABSTRACT. The main purpose of this paper is to prove that there is only one prime among the symmetric sequence. This solved the problem 17 of Professor F.Smarandache in [1].

1. INTRODUCTION

For any positive integer n, we define the symmetric sequence $\{S(n)\}$ as follows: $S(1) = 1$, $S(2) = 11$, $S(3) = 121$, $S(4) = 1221$, $S(5) = 12321$, $S(6) = 123321$, $S(7) = 1234321$, $S(8) = 12344321$, $\cdots\cdots$. In problem 17 of [1], Professor F.Smarandache asked us to solve such a problem: How many primes are there among these numbers? This problem is interesting, because it can help us to find some new symmetric primes. In this paper, we shall study this problem, and give an exact answer. That is, we shall prove the following conclusion:

Theorem. *For any positive integer $n \geq 2$, we have the decomposition*

$$123\cdots(n-1)nn(n-1)\cdots321 = \overbrace{11\cdots1}^{n} \times \overbrace{11\cdots1}^{n+1};$$

$$123\cdots(n-1)n(n-1)\cdots321 = \overbrace{11\cdots1}^{n} \times \overbrace{11\cdots1}^{n}.$$

From this theorem we may immediately deduce the following two corollaries:

Corollary 1. *There is only one prime among the symmetric sequence. That is, $S(2) = 11$.*

Corollary 2. *For any positive integer n, $S(2n-1)$ is a perfect square number. That is,*

$$S(2n-1) = 123\cdots(n-1)n(n-1)\cdots321$$
$$= \overbrace{11\cdots1}^{n} \times \overbrace{11\cdots1}^{n}.$$

Key words and phrases. The symmetric sequence; Primes; A problem of F.Smarandache.
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2. Proof of the Theorem

In this section, we complete the proof of the theorems. First we let

$$S_1 = \{11, 1221, 123321, \cdots\cdots, 123\cdots(n-1)nn(n-1)\cdots 321, \cdots\cdots,\}, \text{ for } n \leq 9,$$

and

$$S_2 = \{1, 121, 12321, \cdots\cdots, 123\cdots(n-1)n(n-1)\cdots 321, \cdots\cdots,\}, \text{ for } n \leq 9.$$

Then it is clear that

$$\{S(n)\} = S_1 \bigcup S_2.$$

For any positive integer $m \in \{S(n)\}$, we have $m \in S_1$ or $m \in S_2$. If $m \in S_1$, then there exists a positive integer n such that $m = 123\cdots(n-1)nn(n-1)\cdots 321$. So that

$$\begin{aligned} m &= 10^{2n-1} + 2 \times 10^{2n-2} + \cdots + n \times 10^n \\ &\quad + n \times 10^{n-1} + (n-1) \times 10^{n-2} + \cdots 2 \times 10 + 1 \\ &= \left[10^{2n-1} + 2 \times 10^{2n-2} + \cdots + n) \times 10^n\right] \\ &\quad + \left[n \times 10^{n-1} + (n-) \times 10^{n-2} + \cdots 2 \times 10 + 1\right] \\ &\equiv S_{11} + S_{12}. \end{aligned} \tag{1}$$

Now we compute S_{11} and S_{12} in (1) respectively. Note that

$$\begin{aligned} 9S_{11} &= 10S_{11} - S_{11} = 10^{2n} + 2 \times 10^{2n-1} + \cdots n \times 10^{n+1} \\ &\quad - 10^{2n-1} - 2 \times 10^{2n-2} - \cdots - n \times 10^n \\ &= 10^{2n} + 10^{2n-1} + 10^{2n-2} + \cdots + 10^{n+1} - n \times 10^n \\ &= 10^{n+1} \times \frac{10^n - 1}{9} - n \times 10^n \end{aligned}$$

and

$$\begin{aligned} 9S_{12} &= 10S_{12} - S_{12} = n \times 10^n + (n-1) \times 10^{n-1} + \cdots 2 \times 10^2 + 10 \\ &\quad - n \times 10^{n-1} - (n-1) \times 10^{n-2} - \cdots 2 \times 10 - 1 \\ &= n \times 10^n - 10^{n-1} - 10^{n-2} - \cdots 10 - 1 \\ &= n \times 10^n - \frac{10^n - 1}{9}. \end{aligned}$$

So that we have

$$S_{11} = \frac{1}{81} \times \left[10^{2n+1} - 9n \times 10^n - 10^{n+1}\right] \tag{2}$$

and

$$S_{12} = \frac{1}{81}\left[9n \times 10^n - 10^n + 1\right]. \tag{3}$$

Combining (1), (2) and (3) we have

$$\begin{aligned} m &= S_{11} + S_{12} \\ &= \frac{1}{81} \times \left[10^{2n+1} - 9n \times 10^n - 10^{n+1}\right] + \frac{1}{81}\left[9n \times 10^n - 10^n + 1\right] \\ &= \frac{1}{81}\left(10^{2n+1} - 10^{n+1} - 10^n + 1\right) \\ &= \frac{1}{81}\left(10^n - 1\right)\left(10^{n+1} - 1\right) \\ &= \overbrace{11\cdots1}^{n} \times \overbrace{11\cdots1}^{n+1}. \end{aligned} \tag{4}$$

If $m \in S_2$, then there exists a positive integer n such that

$$m = 123\cdots(n-1)n(n-1)\cdots321.$$

Similarly, we also have the identity

$$\begin{aligned} m &= 10^{2n-2} + 2 \times 10^{2n-3} + \cdots + n \times 10^{n-1} \\ &\quad + (n-1) \times 10^{n-2} + (n-2) \times 10^{n-3} + \cdots 2 \times 10 + 1 \\ &= \frac{1}{81}\left(10^{2n} - 10^n - 9n \times 10^{n-1}\right) + \frac{1}{81}\left(9n \times 10^{n-1} - 10^n + 1\right) \\ &= \left[\frac{10^n - 1}{9}\right]^2 = \overbrace{11\cdots1}^{n} \times \overbrace{11\cdots1}^{n}. \end{aligned} \tag{5}$$

Now the theorem 1 follows from (4) and (5).

From theorem 1 we know that $S(n)$ is a composite number, if $n \geq 3$, (and $n \leq 9$.) Note that $S(1) = 1$ and $S(2) = 11$ (a prime), we may immediately deduce the theorem 2. This completes the proof of the theorems.

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ON THE PERMUTATION SEQUENCE AND ITS SOME PROPERTIES*

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ABSTRACT. The main purpose of this paper is to prove that there is no any perfect power among the permutation sequence: 12, 1342, 135642, 13578642, 13579108642, ·····. This answered the question 20 of F.Smarandach in [1].

for $n \le 9$ partially

1. INTRODUCTION

For any positive integer n, we define the permutation sequence $\{P(n)\}$ as follows: $P(1) = 12$, $P(2) = 1342$, $P(3) = 135642$, $P(4) = 13578642$, $P(5) = 13579108642$, ·····, $P(n) = 135\cdots(2n-1)(2n)(2n-2)\cdots 42$, ·····,. In problem 20 of [1], Professor F.Smarandach asked us to answer such a question: Is there any perfect power among these numbers? Conjecture: no! This problem is interesting, because it can help us to find some new properties of permutation sequence. In this paper, we shall study the properties of the permutation sequence $P(n)$, and proved that the F.Smarandach conjecture is true. This solved the problem 20 of [1], and more, we also obtained some new divisible properties of $P(n)$. That is, we shall prove the following conclusion:

Theorem. *There is no any perfect power among permutation sequence, and*

$$P(n) = \frac{1}{81}\left(11\cdot 10^{2n} - 13\cdot 10^n + 2\right) = \overbrace{11\cdots 1}^{n} \times 1\overbrace{22\cdots 2}^{n}, \text{ for } n \le 9.$$

2. PROOF OF THE THEOREM

In this section, we complete the proof of the Theorem. First for any positive integer n, we have

$$\begin{aligned} P(n) &= 10^{2n-1} + 3\times 10^{2n-2} + \cdots + (2n-1)\times 10^n \\ &\quad + 2n\times 10^{n-1} + (2n-2)\times 10^{n-2} + \cdots 4\times 10 + 2 \\ &= \left[10^{2n-1} + 3\times 10^{2n-2} + \cdots + (2n-1)\times 10^n\right] \\ &\quad + \left[2n\times 10^{n-1} + (2n-2)\times 10^{n-2} + \cdots 4\times 10 + 2\right] \end{aligned}$$

Key words and phrases. Permutation sequence; Perfect power; A problem of F.Smarandach.
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$$(1) \qquad \equiv S_1 + S_2.$$

Now we compute S_1 and S_2 in (1) respectively. Note that

$$\begin{aligned} 9S_1 = 10S_1 - S_1 &= 10^{2n} + 3 \times 10^{2n-1} + \cdots (2n-1) \times 10^{n+1} \\ &\quad - 10^{2n-1} - 3 \times 10^{2n-2} - \cdots - (2n-1) \times 10^n \\ &= 10^{2n} + 2 \times 10^{2n-1} + 2 \times 10^{2n-2} + \cdots + 2 \times 10^{n+1} - (2n-1) \times 10^n \\ &= 10^{2n} + 2 \times 10^{n+1} \times \frac{10^{n-1} - 1}{9} - (2n-1) \times 10^n \end{aligned}$$

and

$$\begin{aligned} 9S_2 = 10S_2 - S_2 &= 2n \times 10^n + (2n-2) \times 10^{n-1} + \cdots 4 \times 10^2 + 2 \times 10 \\ &\quad - 2n \times 10^{n-1} - (2n-2) \times 10^{n-2} - \cdots 4 \times 10 - 2 \\ &= 2n \times 10^n - 2 \times 10^{n-1} - 2 \times 10^{n-2} - \cdots 2 \times 10 - 2 \\ &= 2n \times 10^n - 2 \times \frac{10^n - 1}{9}. \end{aligned}$$

So that

$$(2) \qquad S_1 = \frac{1}{81} \times [11 \times 10^{2n} - 18n \times 10^n - 11 \times 10^n]$$

and

$$(3) \qquad S_2 = \frac{1}{81} [18n \times 10^n - 2 \times 10^n + 2].$$

Thus combining (1), (2) and (3) we have

$$\begin{aligned} P(n) = S_1 + S_2 &= \frac{1}{81} \times [11 \times 10^{2n} - 18n \times 10^n - 11 \times 10^n] \\ &\quad + \frac{1}{81} [18n \times 10^n - 2 \times 10^n + 2] \\ (4) \qquad &= \frac{1}{81} (11 \cdot 10^{2n} - 13 \cdot 10^n + 2) = \overbrace{11\cdots1}^{n} \times 1 \overbrace{22\cdots2}^{n}. \end{aligned}$$

From (4) we can easily find that $2 \mid P(n)$, but $4 \nmid P(n)$, if $n \geq 2$, So that $P(n)$ can not be a perfect power, if $n \geq 2$. In fact, if we assume $P(n)$ be a perfect power, then $P(n) = m^k$, for some positive integer $m \geq 2$ and $k \geq 2$. Since $2 \mid P(n)$, so that m must be an even number. Thus we have $4 \mid P(n)$. This contradiction with $4 \nmid P(n)$, if $n \geq 2$. Note that $P(1)$ is not a perfect power, so that $P(n)$ can be a perfect power for all $n \geq 1$, This completes the proof of the Theorem.

References

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A NUMBER THEORETIC FUNCTION AND ITS MEAN VALUE PROPERTY*

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ABSTRACT. Let p be a prime, n be any positive integer, $\alpha(n,p)$ denotes the power of p in the factorization of $n!$. In this paper, we study the asymptotic properties of the mean value $\sum_{p\le n}\alpha(n,p)$, and give an interesting asymptotic formula for it.

1. INTRODUCTION

Let p be a prime, $e_p(n)$ denotes the largest exponent (of power p) which divides n, $\alpha(n,p)=\sum_{k\le n}e_p(k)$. In problem 68 of [1], Professor F.Smarandach asked us to study the properties of the sequences $e_p(n)$. This problem is interesting because there are close relations between $e_p(n)$ and the factorization of $n!$. In fact, $\alpha(n,p)$ is the power of p in the factorization of $n!$. In this paper, we use the elementary methods to study the asymptotic properties of the mean value $\sum_{p\le n}\alpha(n,p)$, and give an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. *For any prime p and any fixed positive integer n, we have the asymptotic formula*

$$\sum_{p\le n}\alpha(n,p)=n\ln\ln n+cn+c_1\frac{n}{\ln n}+c_2\frac{n}{\ln^2 n}+\cdots+c_k\frac{n}{\ln^k n}+O\left(\frac{n}{\ln^{k+1}n}\right).$$

where k is any fixed positive integer, c_i $(i=1,2,\cdots,)$ are some computable constants.

2. PROOF OF THE THEOREM

In this section, we complete the proof of the Theorem. First for any prime p and any fixed positive integer n, we let $a(n,p)$ denote the sum of the base p digits of n. That is, if $n=a_1p^{\alpha_1}+a_2p^{\alpha_2}+\cdots+a_sp^{\alpha_s}$ with $\alpha_s>\alpha_{s-1}>\cdots>\alpha_1\ge 0$, where $1\le a_i\le p-1$, $i=1,2,\cdots,s$, then $a(n,p)=\sum_{i=1}^{s}a_i$, and for this number theoretic function, we have the following two simple Lemmas:

Key words and phrases. A new number theoretic function; Mean value; Asymptotic formula.
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Lemma 1. *For any integer $n \geq 1$, we have the identity*

$$\alpha_p(n) \equiv \alpha(n) \equiv \sum_{i=1}^{+\infty} \left[\frac{n}{p^i}\right] = \frac{1}{p-1}(n - a(n,p)),$$

where $[x]$ denotes the greatest integer not exceeding x.

Proof. From the properties of $[x]$ we know that

$$\left[\frac{n}{p^i}\right] = \left[\frac{a_1 p^{\alpha_1} + a_2 p^{\alpha_2} + \cdots + a_s p^{\alpha_s}}{p^i}\right]$$
$$= \begin{cases} \sum_{j=k}^{s} a_j p^{\alpha_j - i}, & \text{if } \alpha_{k-1} < i \leq \alpha_k \\ 0, & \text{if } i > \alpha_s. \end{cases}$$

So from this formula we have

$$\alpha(n) \equiv \sum_{i=1}^{+\infty} \left[\frac{n}{p^i}\right] = \sum_{i=1}^{+\infty} \left[\frac{a_1 p^{\alpha_1} + a_2 p^{\alpha_2} + \cdots + a_s p^{\alpha_s}}{p^i}\right]$$
$$= \sum_{j=1}^{s} \sum_{k=1}^{\alpha_j} a_j p^{\alpha_j - k} = \sum_{j=1}^{s} a_j \left(1 + p + p^2 + \cdots + p^{\alpha_j - 1}\right)$$
$$= \sum_{j=1}^{s} a_j \cdot \frac{p^{\alpha_j} - 1}{p-1} = \frac{1}{p-1} \sum_{j=1}^{s} (a_j p^{\alpha_j} - a_j)$$
$$= \frac{1}{p-1}(n - a(n,p)).$$

This completes the proof of Lemma 1.

Lemma 2. *For any positive integer n, we have the estimate*

$$a(n,p) \leq \frac{p-1}{\ln p} \ln n.$$

Proof. Let $n = a_1 p^{\alpha_1} + a_2 p^{\alpha_2} + \cdots + a_s p^{\alpha_s}$ with $\alpha_s > \alpha_{s-1} > \cdots > \alpha_1 \geq 0$, where $1 \leq a_i \leq p-1$, $i = 1, 2, \cdots, s$. Then from the definition of $a(n,p)$ we have

$$a(n,p) = \sum_{i=1}^{s} a_i \leq \sum_{i=1}^{s} (p-1) = (p-1)s. \tag{1}$$

On the other hand, using the mathematical induction we can easily get the inequality

$$n = a_1 p^{\alpha_1} + a_2 p^{\alpha_2} + \cdots + a_s p^{\alpha_s} \geq a_s p^s,$$

or

$$s \leq \frac{\ln(n/a_s)}{\ln p} \leq \frac{\ln n}{\ln p}. \tag{2}$$

Combining (1) and (2) we immediately get the estimate

$$a(n,p) \le \frac{p-1}{\ln p}\ln n.$$

This proves Lemma 2.

Now we use Lemma 1 and Lemma 2 to complete the proof of the Theorem. First, we separate the summation in the Theorem into two parts.

$$\sum_{p\le n}\alpha(n,p) = \sum_{p\le\sqrt{n}}\alpha(n,p) + \sum_{\sqrt{n}<p\le n}\alpha(n,p). \tag{3}$$

For the first part, from Lemma 1 we have

$$\begin{aligned}
\sum_{p\le\sqrt{n}}\alpha(n,p) &= \sum_{p\le\sqrt{n}}\frac{1}{p-1}(n-a(n,p)) \\
&= n\sum_{p\le\sqrt{n}}\left(\frac{1}{p}+\frac{1}{p(p-1)}\right) - \sum_{p\le\sqrt{n}}\frac{a(n,p)}{p-1} \\
&= n\left(\sum_{p\le\sqrt{n}}\frac{1}{p}+\sum_{p}\frac{1}{p(p-1)}+O\left(\sum_{m>\sqrt{n}}\frac{1}{m^2}\right)\right) - \sum_{p\le\sqrt{n}}\frac{a(n,p)}{p-1} \\
&= n\left(\int_{\frac{3}{2}}^{\sqrt{n}}\frac{1}{x}d\pi(x)+A+O\left(\frac{1}{\sqrt{n}}\right)\right) - \sum_{p\le\sqrt{n}}\frac{a(n,p)}{p-1}. \qquad (4)
\end{aligned}$$

where $\pi(x)$ denotes the number of all prime not exceeding x. For $\pi(x)$, we have the asymptotic formula

$$\pi(x) = \frac{x}{\ln x}+a_2\frac{x}{\ln^2 x}+\cdots+a_k\frac{x}{\ln^k x}+O\left(\frac{x}{\ln^{k+1}x}\right) \tag{5}$$

and

$$\begin{aligned}
\int_{\frac{3}{2}}^{\sqrt{n}}\frac{1}{x}d\pi(x) &= \frac{\pi(\sqrt{n})}{\sqrt{n}}+\int_{\frac{3}{2}}^{\sqrt{n}}\frac{\pi(x)}{x^2}dx \\
&= \frac{1}{\ln\sqrt{n}}+\frac{a_2}{\ln^2\sqrt{n}}+\cdots+\frac{a_k}{\ln^k\sqrt{n}}+O\left(\frac{1}{\ln^{k+1}\sqrt{n}}\right)+\int_{\frac{3}{2}}^{\sqrt{n}}\frac{1}{x\ln x}dx \\
&\quad + a_2\int_{\frac{3}{2}}^{\sqrt{n}}\frac{1}{x\ln^2 x}dx+\cdots+a_{k+1}\int_{\frac{3}{2}}^{\sqrt{n}}\frac{1}{x\ln^{k+1}x}dx+O\left(\frac{1}{\ln^{k+1}n}\right) \\
&= \frac{a_{11}}{\ln n}+\frac{a_{12}}{\ln^2 n}+\cdots+\frac{a_{1k}}{\ln^k n}+\ln\ln n+B+\frac{a_{21}}{\ln n}+\frac{a_{22}}{\ln^2 n} \\
&\quad +\cdots+\frac{a_{2k}}{\ln^k n}+O\left(\frac{1}{\ln^{k+1}n}\right) \\
&= \ln\ln n+B+\frac{a_{31}}{\ln n}+\frac{a_{32}}{\ln^2 n}+\cdots+\frac{a_{3k}}{\ln^k n}+O\left(\frac{1}{\ln^{k+1}n}\right). \qquad (6)
\end{aligned}$$

From Lemma 2 we have

$$(7)\qquad \sum_{p\le\sqrt{n}}\frac{a(n,p)}{p-1}\le\sum_{p\le\sqrt{n}}\frac{\ln n}{\ln p}=\ln n\sum_{p\le\sqrt{n}}\frac{1}{\ln p}\le\ln n\sum_{p\le\sqrt{n}}1\le\sqrt{n}\ln n.$$

Combining (4), (6) and (7) we obtain

$$\begin{aligned}\sum_{p\le\sqrt{n}}\alpha(n,p)&=n\ln\ln n+c_0n+a_{31}\frac{n}{\ln n}+a_{32}\frac{n}{\ln^2 n}\\ (8)\qquad &\quad+\cdots+a_{3k}\frac{n}{\ln^k n}+O\left(\frac{n}{\ln^{k+1}n}\right).\end{aligned}$$

For the second part, we have

$$\begin{aligned}\sum_{\sqrt{n}<p\le n}\alpha(n,p)&=\sum_{\sqrt{n}<p\le n}\sum_{i=1}^{+\infty}\left[\frac{n}{p^i}\right]=\sum_{\sqrt{n}<p\le n}\left[\frac{n}{p}\right]=\sum_{\sqrt{n}<p\le n}\sum_{m\le\frac{n}{p}}1\\ &=\sum_{m\le\sqrt{n}}\sum_{\sqrt{n}<p\le\frac{n}{m}}1=\sum_{m\le\sqrt{n}}\left(\pi\left(\frac{n}{m}\right)-\pi(\sqrt{n})\right)\\ (9)\qquad &=\sum_{m\le\sqrt{n}}\pi\left(\frac{n}{m}\right)-[\sqrt{n}]\pi(\sqrt{n}).\end{aligned}$$

Applying Euler's summation formula (see [2] Theorem 3.1) and the expansion into power-series we have

$$\begin{aligned}\sum_{m\le\sqrt{n}}\frac{1}{m(\ln n-\ln m)^r}&=\sum_{m\le\sqrt{n}}\frac{1}{m\ln^r n(1-\frac{\ln m}{\ln n})^r}\\ &=\sum_{s=0}^{+\infty}\sum_{m\le\sqrt{n}}\frac{\binom{r-1+s}{r-1}\ln^s m}{m\ln^{s+r}n}\\ &=\sum_{s=0}^{+\infty}\binom{r-1+s}{r-1}\left(\sum_{m\le\sqrt{n}}\frac{\ln^s m}{m\ln^{s+r}n}\right)\\ &=\sum_{s=0}^{+\infty}\frac{\binom{r-1+s}{r-1}}{\ln^{s+r}n}\left(\frac{\ln^{s+1}n}{(s+1)2^{s+1}}+d_{s+1}+O\left(\frac{\ln^s n}{2^s\sqrt{n}}\right)\right)\\ &=\sum_{i=r-1}^{k}\frac{d_{1i}}{\ln^i n}+O\left(\frac{\ln^s n}{\sqrt{n}}\right).\end{aligned}$$

From this and (5) we get

$$\sum_{m\le\sqrt{n}} \pi\left(\frac{n}{m}\right)$$

$$= \sum_{m\le\sqrt{n}} \left(\frac{\frac{n}{m}}{\ln\left(\frac{n}{m}\right)} + a_2 \frac{\frac{n}{m}}{\ln^2\left(\frac{n}{m}\right)} + \cdots + a_{k+1} \frac{\frac{n}{m}}{\ln^{k+1}\left(\frac{n}{m}\right)} + O\left(\frac{\frac{n}{m}}{\ln^{k+2}\left(\frac{n}{m}\right)} \right) \right)$$

$$= n \sum_{m\le\sqrt{n}} \left(\frac{1}{m(\ln n - \ln m)} + a_2 \frac{1}{m(\ln n - \ln m)^2} + \cdots \right.$$

$$\left. + a_{k+1} \frac{1}{m(\ln n - \ln m)^{k+1}} + O\left(\frac{1}{m(\ln n - \ln m)^{k+2}} \right) \right)$$

$$= n \left(b_0 + \frac{b_1}{\ln n} + \frac{b_2}{\ln^2 n} + \cdots + \frac{b_k}{\ln^k n} + O\left(\frac{1}{\ln^{k+1} n} \right) \right)$$

(10)

$$= b_0 n + b_1 \frac{n}{\ln n} + b_2 \frac{n}{\ln^2 n} + \cdots + b_k \frac{n}{\ln^k n} + O\left(\frac{n}{\ln^{k+1} n} \right)$$

and

$$[\sqrt{n}]\pi(\sqrt{n}) = \frac{n}{\ln\sqrt{n}} + a_2 \frac{n}{\ln^2\sqrt{n}} + \cdots + a_k \frac{n}{\ln^k\sqrt{n}} + O\left(\frac{n}{\ln^{k+1}\sqrt{n}} \right)$$

(11)

$$= a_{41} \frac{n}{\ln n} + a_{42} \frac{n}{\ln^2 n} + \cdots + a_{4k} \frac{n}{\ln^k n} + O\left(\frac{n}{\ln^{k+1} n} \right).$$

Combining (9), (10) and (11) we have

$$(12) \quad \sum_{\sqrt{n}<p\le n} \alpha(n,p) = b_0 n + a_{51} \frac{n}{\ln n} + a_{52} \frac{n}{\ln^2 n} + \cdots + a_{5k} \frac{n}{\ln^k n} + O\left(\frac{n}{\ln^{k+1} n} \right).$$

From (3), (8) and (12) we obtain the asymptotic formula

$$\sum_{p\le n} \alpha(n,p) = n \ln\ln n + cn + c_1 \frac{n}{\ln n} + c_2 \frac{n}{\ln^2 n} + \cdots + c_k \frac{n}{\ln^k n} + O\left(\frac{n}{\ln^{k+1} n} \right).$$

This completes the proof of the Theorem.

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An introduction to the Smarandache Square Complementary function

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Abstract

In this paper the main properties of Smarandache Square Complementary function has been analysed. Several problems still unsolved are reported too.

The Smarandache square complementary function is defined as [4],[5]:

$$Ssc(n)=m$$

where m is the smallest value such that $m \cdot n$ is a perfect square.

Example: for n=8, m is equal 2 because this is the least value such that $m \cdot n$ is a perfect square.

The first 100 values of Ssc(n) function follows:

n	Ssc(n)	n	Ssc(n)	n	Ssc(n)	n	Ssc(n)
1	1	26	26	51	51	76	19
2	2	27	3	52	13	77	77
3	3	28	7	53	53	78	78
4	1	29	29	54	6	79	79
5	5	30	30	55	55	80	5
6	6	31	31	56	14	81	1
7	7	32	2	57	57	82	82
8	2	33	33	58	58	83	83
9	1	34	34	59	59	84	21
10	10	35	35	60	15	85	85
11	11	36	1	61	61	86	86
12	3	37	37	62	62	87	87
13	13	38	38	63	7	88	22
14	14	39	39	64	1	89	89
15	15	40	10	65	65	90	10
16	1	41	41	66	66	91	91
17	17	42	42	67	67	92	23
18	2	43	43	68	17	93	93
19	19	44	11	69	69	94	94
20	5	45	5	70	70	95	95
21	21	46	46	71	71	96	6
22	22	47	47	72	2	97	97
23	23	48	3	73	73	98	2
24	6	49	1	74	74	99	11
25	1	50	2	75	3	100	1

Let's start to explore some properties of this function.

Theorem 1: $Ssc(n^2)=1$ *where n=1,2,3,4...*

In fact if $k=n^2$ is a perfect square by definition the smallest integer m such that $m\cdot k$ is a perfect square is m=1.

Theorem 2: *Ssc(p)=p where p is any prime number*

In fact in this case the smallest m such that $m\cdot p$ is a perfect square can be only m=p.

Theorem 3: $Ssc(p^n)=\begin{cases} 1 & \text{if } n \text{ is even} \\ p & \text{if } n \text{ is odd} \end{cases}$ *where p is any prime number.*

First of all let's analyse the even case. We can write:

$$p^n = p^2\cdot p^2\cdot\ldots\ldots\cdot p^2 = \left|p^{\frac{n}{2}}\right|^2$$ and then the smallest m such that $p^n\cdot m$ is a perfect square is 1.

Let's suppose now that n is odd. We can write:

$$p^n = p^2\cdot p^2\cdot\ldots\ldots\cdot p^2\cdot p = \left|p^{\left\lfloor\frac{n}{2}\right\rfloor}\right|^2\cdot p = p^{2\left\lfloor\frac{n}{2}\right\rfloor}\cdot p$$

and then the smallest integer m such that $p^n\cdot m$ is a perfect square is given by m=p.

Theorem 4: $Ssc(p^a\cdot q^b\cdot s^c\cdot\ldots\ldots\cdot t^x)=p^{odd(a)}\cdot q^{odd(b)}\cdot s^{odd(c)}\cdot\ldots\cdot t^{odd(x)}$ *where p,q,s,....t are distinct primes and the odd function is defined as:*

$$odd(n)=\begin{cases} 1 & \text{if n is odd} \\ 0 & \text{if n is even} \end{cases}$$

Direct consequence of theorem 3.

Theorem 5: *The Ssc(n) function is multiplicative, i.e. if (n,m)=1 then* $Ssc(n \cdot m) = Ssc(n) \cdot Ssc(m)$

Without loss of generality let's suppose that $n = p^a \cdot q^b$ and $m = s^c \cdot t^d$ where p, q, s, t are distinct primes. Then:

$$Ssc(n \cdot m) = Ssc(p^a \cdot q^b \cdot s^c \cdot t^d) = p^{odd(a)} \cdot q^{odd(b)} \cdot s^{odd(c)} \cdot t^{odd(d)}$$

according to the theorem 4.

On the contrary:

$$Ssc(n) = Ssc(p^a \cdot q^b) = p^{odd(a)} \cdot q^{odd(b)}$$

$$Ssc(m) = Ssc(s^c \cdot t^d) = s^{odd(c)} \cdot t^{odd(d)}$$

This implies that: $Ssc(n \cdot m) = Ssc(n) \cdot Ssc(m)$ qed

Theorem 6: *If* $n = p^a \cdot q^b \cdot \ldots \cdot p^s$ *then* $Ssc(n) = Ssc(p^a) \cdot Ssc(p^b) \cdot \ldots \cdot Ssc(p^s)$ *where p is any prime number.*

According to the theorem 4:

$$Ssc(n) = p^{odd(a)} \cdot p^{odd(b)} \cdot \ldots \cdot p^{odd(s)}$$

and:

$$Ssc(p^a) = p^{odd(a)}$$
$$Ssc(p^b) = p^{odd(b)}$$

and so on. Then:

$$Ssc(n) = Ssc(p^a) \cdot Ssc(p^b) \cdot \ldots \cdot Ssc(p^s)$$ qed

Theorem 7: *Ssc(n)=n if n is squarefree, that is if the prime factors of n are all distinct. All prime numbers, of course are trivially squarefree [3].*

Without loss of generality let's suppose that $n = p \cdot q$ where p and q are two distinct primes.

According to the theorems 5 and 3:

$Ssc(n) = Ssc(p \cdot q) = Ssc(p) \cdot Ssc(q) = p \cdot q = n$ qed

Theorem 8: *The Ssc(n) function is not additive.*:

In fact for example: Ssc(3+4)=Ssc(7)=7<>Ssc(3)+Ssc(4)=3+1=4

Anyway we can find numbers m and n such that the function Ssc(n) is additive. In fact if:

m and n are squarefree
k=m+n is squarefree.

then Ssc(n) is additive.
In fact in this case Ssc(m+n)=Ssc(k)=k=m+n and Ssc(m)=m Ssc(n)=n according to theorem 7.

Theorem 9: $\sum_{n=1}^{\infty} \frac{1}{Ssc(n)}$ *diverges*

In fact:

$$\sum_{n=1}^{\infty} \frac{1}{Ssc(n)} > \sum_{p=2}^{\infty} \frac{1}{Ssc(p)} = \sum_{p=2}^{\infty} \frac{1}{p}$$ where p is any prime number.

So the sum of inverese of Ssc(n) function diverges due to the well known divergence of series [3]:

$$\sum_{p=2}^{\infty} \frac{1}{p}$$

Theorem 10: *Ssc(n)>0 where n=1,2,3,4 ...*

This theorem is a direct consequence of Ssc(n) function definition. In fact for any n the smallest m such that $m \cdot n$ is a perfect square cannot be equal to zero otherwise $m \cdot n$=0 and zero is not a perfect square.

Theorem 11: $\sum_{n=1}^{\infty} \frac{Ssc(n)}{n}$ *diverges*

In fact being $Ssc(n) \geq 1$ this implies that:

$$\sum_{n=1}^{\infty} \frac{Ssc(n)}{n} > \sum_{n=1}^{\infty} \frac{1}{n}$$

and as known the sum of reciprocal of integers diverges. [3]

Theorem 12: $Ssc(n) \leq n$

Direct consequence of theorem 4.

Theorem 13: *The range of Ssc(n) function is the set of squarefree numbers.*

According to the theorem 4 for any integer n the function Ssc(n) generates a squarefree number.

Theorem 14: $0 < \frac{Ssc(n)}{n} \leq 1$ *for n>=1*

Direct consequence of theorems 12 and 10.

Theorem 15: $\frac{Ssc(n)}{n}$ *is not distributed uniformly in the interval]0,1]*

If n is squarefree then Ssc(n)=n that implies $\frac{Ssc(n)}{n} = 1$

If n is not squarefree let's suppose without loss of generality that $n = p^a \cdot q^b$ where p and q are primes.

Then:

$$\frac{Ssc(n)}{n} = \frac{Ssc(p^a) \cdot Ssc(p^b)}{p^a \cdot q^b}$$

We can have 4 different cases.

1) a even and b even

$$\frac{Ssc(n)}{n} = \frac{Ssc(p^a)\cdot Ssc(p^b)}{p^a\cdot q^b} = \frac{1}{p^a\cdot q^b} \le \frac{1}{4}$$

2) a odd and b odd

$$\frac{Ssc(n)}{n} = \frac{Ssc(p^a)\cdot Ssc(p^b)}{p^a\cdot q^b} = \frac{p\cdot q}{p^a\cdot q^b} = \frac{1}{p^{a-1}\cdot q^{b-1}} \le \frac{1}{4}$$

3) a odd and b even

$$\frac{Ssc(n)}{n} = \frac{Ssc(p^a)\cdot Ssc(p^b)}{p^a\cdot q^b} = \frac{p\cdot 1}{p^a\cdot q^b} = \frac{1}{p^{a-1}\cdot q^b} \le \frac{1}{4}$$

4) a even and b odd

Analogously to the case 3 .

This prove the theorem because we don't have any point of Ssc(n) function in the interval]1/4,1[

Theorem 16: *For any arbitrary real number* $\varepsilon > 0$, *there is some number n>=1 such that:*

$$\frac{Ssc(n)}{n} < \varepsilon$$

Without loss of generality let's suppose that $q = p_1 \cdot p_2$ where p_1 and p_2 are primes such that $\frac{1}{q} < \varepsilon$ and ε is any real number grater than zero. Now take a number n such that:

$$n = p_1^{a_1} \cdot p_2^{a_2}$$

For a_1 and a_2 odd:

$$\frac{Ssc(n)}{n} = \frac{p_1 \cdot p_2}{p_1^{a_1} \cdot p_2^{a_2}} = \frac{1}{p_1^{a_1-1} \cdot p_2^{a_2-1}} < \frac{1}{p_1 \cdot p_2} < \varepsilon$$

For a_1 and a_2 even:

$$\frac{Ssc(n)}{n} = \frac{1}{p_1^{a_1} \cdot p_2^{a_2}} < \frac{1}{p_1 \cdot p_2} < \varepsilon$$

For a_1 odd and a_2 even (or viceversa):

$$\frac{Ssc(n)}{n} = \frac{p_1}{p_1^{a_1} \cdot p_2^{a_2}} = \frac{1}{p_1^{a_1-1} \cdot p_2^{a_2}} < \frac{1}{p_1 \cdot p_2} < \varepsilon$$

Theorem 17: $Ssc(p_k\#) = p_k\#$ where $p_k\#$ is the product of first k primes (primorial) [3].

The theorem is a direct consequence of theorem 7 being $p_k\#$ a squarefree number.

Theorem 18: *The equation* $\frac{Ssc(n)}{n} = 1$ *has an infinite number of solutions.*

The theorem is a direct consequence of theorem 2 and the well-known fact that there is an infinite number of prime numbers [6]

Theorem 19: *The repeated iteration of the Ssc(n) function will terminate always in a fixed point (see [3] for definition of a fixed point).*

According to the theorem 13 the application of Scc function to any n will produce always a squarefree number and according to the theorem 7 the repeated application of Ssc to this squarefree number will produce always the same number.

Theorem 20: *The diophantine equation Ssc(n)=Ssc(n+1) has no solutions.*

We must distinguish three cases:

1) n and n+1 squarefree
2) n and n+1 not squareefree
3) n squarefree and n+1 no squarefree and viceversa

Case 1. According to the theorem 7 Ssc(n)=n and Ssc(n+1)=n+1 that implies that Ssc(n)<>Ssc(n+1)

Case 2. Withou loss of generality let's suppose that:

$$n = p^a \cdot q^b$$
$$n+1 = p^a \cdot q^b + 1 = s^c \cdot t^d$$

where p,q,s and t are distinct primes.

According to the theorem 4:

$$Ssc(n) = Ssc(p^a \cdot q^b) = p^{odd(a)} \cdot q^{odd(b)}$$
$$Ssc(n+1) = Ssc(s^c \cdot t^d) = s^{odd(c)} \cdot t^{odd(d)}$$

and then Ssc(n)<>Ssc(n+1)

Case 3. Without loss of generality let's suppose that $n = p \cdot q$. Then:

$$Ssc(n) = Ssc(p \cdot q) = p \cdot q$$
$$Ssc(n+1) = Ssc(p \cdot q + 1) = Ssc(s^a \cdot t^b) = s^{odd(a)} \cdot t^{odd(b)}$$

supposing that $n+1 = p \cdot q + 1 = s^a \cdot t^b$

This prove completely the theorem.

Theorem 21: $\sum_{k=1}^{N} Ssc(k) > \frac{6 \cdot N}{\pi^2}$ *for any positive integer N.*

The theorem is very easy to prove. In fact the sum of first N values of Ssc function can be separated into two parts:

$$\sum_{k_1=1}^{N} Ssc(k_1) + \sum_{k_2=1}^{N} Ssc(k_2)$$

where the first sum extend over all k_1 squarefree numbers and the second one over all k_2 not squarefree numbers.
According to the Hardy and Wright result [3], the asymptotic number Q(n) of squarefree numbers $\leq N$ is given by:

$$Q(N) \approx \frac{6 \cdot N}{\pi^2}$$

and then:

$$\sum_{k=1}^{N} Ssc(k) = \sum_{k_1=1}^{N} Ssc(k_1) + \sum_{k_2=1}^{N} Ssc(k_2) > \frac{6 \cdot N}{\pi^2}$$

because according to the theorem 7, $Ssc(k_1) = k_1$ and the sum of first N squarefree numbers is always greater or equal to the number Q(N) of squarefree numbers $\leq N$, namely:

$$\sum_{k_1=1}^{N} k_1 \geq Q(N)$$

Theorem 22: $\sum_{k=1}^{N} Ssc(k) > \frac{N^2}{2 \cdot \ln(N)}$ *for any positive integer N.*

In fact:

$$\sum_{k=1}^{N} Ssc(k) = \sum_{k'=1}^{N} Ssc(k') + \sum_{p=2}^{N} Ssc(p) > \sum_{p=2}^{N} Ssc(p)$$

because by theorem 2, Ssc(p)=p. But according to the result of Bach and Shallit [3], the sum of first N primes is asymptotically equal to:

$$\frac{N^2}{2\cdot \ln(N)}$$

and this completes the proof.

Theorem 23: *The diophantine equations* $\frac{Ssc(n+1)}{Ssc(n)}=k$ *and* $\frac{Ssc(n)}{Ssc(n+1)}=k$ *where k is any integer number have an infinite number of solutions.*

Let's suppose that n is a perfect square. In this case according to the theorem 1 we have:

$$\frac{Ssc(n+1)}{Ssc(n)}=Ssc(n+1)=k$$

On the contrary if n+1 is a perfect square then:

$$\frac{Ssc(n)}{Ssc(n+1)}=Ssc(n)=k$$

Problems.

1) Is the difference |Ssc(n+1)-Ssc(n)| bounded or unbounded?

2) Is the Ssc(n) function a Lipschitz function ?
A function is said a Lipschitz function [3] if:

$\frac{|Ssc(m)-Ssc(k)|}{|m-k|}\geq M$ where M is any integer

3) Study the function FSsc(n)=m. Here m is the number of different integers k such that Ssc(k)=n.

4) Solve the equations Ssc(n)=Ssc(n+1)+Ssc(n+2) and Ssc(n)+Ssc(n+1)=Ssc(n+2). Is the number of solutions finite or infinite?

5) Find all the values of n such that $Ssc(n) = Ssc(n+1) \cdot Ssc(n+2)$

6) Solve the equation $Ssc(n) \cdot Ssc(n+1) = Ssc(n+2)$

7) Solve the equation $Ssc(n) \cdot Ssc(n+1) = Ssc(n+2) \cdot Ssc(n+3)$

8) Find all the values of n such that $S(n)^k + Z(n)^k = Ssc(n)^k$ where S(n) is the Smarandache function [1], Z(n) the pseudo-Smarandache funtion [2] and k any integer.

9) Find the smallest k such that between Ssc(n) and Ssc(k+n), for n>1, there is at least a prime.

10) Find all the values of n such that Ssc(Z(n))-Z(Ssc(n))=0 where Z is the Pseudo Smarandache function [2].

11) Study the functions Ssc(Z(n)), Z(Ssc(n)) and Ssc(Z(n))-Z(Ssc(n)).

12) Evaluate $\lim_{k\to\infty} \frac{Ssc(k)}{\theta(k)}$ where $\theta(k) = \sum_{n\le k} \ln(Ssc(n))$

13) Are there m, n, k non-null positive integers for which $Ssc(m \cdot n) = m^k \cdot Ssc(n)$?

14) Study the convergence of the Smarandache Square compolementary harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{Ssc^a(n)}$$

where a>0 and belongs to R

15) Study the convergence of the series:

$$\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{Ssc(x_n)}$$

where x_n is any increasing sequence such that $\lim_{n\to\infty} x_n = \infty$

16) Evaluate:

$$\lim_{n\to\infty} \frac{\sum_{k=2}^{n} \frac{\ln(Ssc(k))}{\ln(k)}}{n}$$

Is this limit convergent to some known mathematical constant?

17) Solve the functional equation:

$$Ssc(n)^r + Ssc(n)^{r-1} + \ldots\ldots + Ssc(n) = n$$

where r is an integer ≥ 2.

18) What about the functional equation:

$$Ssc(n)^r + Ssc(n)^{r-1} + \ldots\ldots + Ssc(n) = k \cdot n$$

where r and k are two integers ≥ 2.

19) Evaluate $\sum_{k=1}^{\infty} (-1)^k \cdot \frac{1}{Ssc(k)}$

20) Evaluate $\frac{\sum_n Ssc(n)^2}{\left|\sum_n Ssc(n)\right|^2}$

21) Evaluate:

$$\lim_{n\to\infty}\left|\sum_n \frac{1}{Ssc(f(n))} - \sum_n \frac{1}{f(Ssc(n))}\right|$$

for f(n) equal to the Smarandache function S(n) [1] and to the Pseudo Smarandache function Z(n) [2].

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ON THE PRIMITIVE NUMBERS OF POWER P AND ITS ASYMPTOTIC PROPERTY*

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ABSTRACT. Let p be a prime, n be any positive integer, $S_p(n)$ denotes the smallest integer such that $S_p(n)!$ is divisible by p^n. In this paper, we study the asymptotic properties of $S_p(n)$, and give an interesting asymptotic formula for it.

1. INTRODUCTION

Let p be a prime, n be any positive integer, $S_p(n)$ denotes the smallest integer such that $S_p(n)!$ is divisible by p^n. For example, $S_3(1) = 3$, $S_3(2) = 6$, $S_3(3) = S_3(4) = 9$, $\cdots\cdots$. In problem 49 of book [1], Professor F.Smarandache ask us to study the properties of the sequence $\{S_p(n)\}$. About this problem, it appears that no one had studied it yet, at least, we have not seen such a paper before. The problem is interesting because it can help us to calculate the Smarandache function. In this paper, we use the elementary methods to study the asymptotic properties of $S_p(n)$, and give an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. *For any fixed prime p and any positive integer n, we have the asymptotic formula*

$$S_p(n) = (p-1)n + O\left(\frac{p}{\ln p} \cdot \ln n\right).$$

From this theorem we may immediately deduce the following:

Corollary. *For any positive integer n, we have the asymptotic formulas*

$$a) \qquad S_2(n) = n + O(\ln n);$$

$$b) \qquad S_3(n) = 2n + O(\ln n).$$

Key words and phrases. F.Smarandache problem; Primitive numbers; Asymptotic formula.
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2. Proof of the Theorem

In this section, we complete the proof of the Theorem. First for any fixed prime p and any positive integer n, we let $a(n,p)$ denote the sum of the base p digits of n. That is, if $n = a_1p^{\alpha_1} + a_2p^{\alpha_2} + \cdots + a_sp^{\alpha_s}$ with $\alpha_s > \alpha_{s-1} > \cdots > \alpha_1 \geq 0$, where $1 \leq a_i \leq p-1$, $i = 1, 2, \cdots, s$, then $a(n,p) = \sum_{i=1}^{s} a_i$, and for this number theoretic function, we have the following two simple Lemmas:

Lemma 1. *For any integer $n \geq 1$, we have the identity*

$$\alpha_p(n) \equiv \alpha(n) \equiv \sum_{i=1}^{+\infty} \left[\frac{n}{p^i}\right] = \frac{1}{p-1}\left(n - a(n,p)\right),$$

where $[x]$ denotes the greatest integer not exceeding x.

Proof. From the properties of $[x]$ we know that

$$\left[\frac{n}{p^i}\right] = \left[\frac{a_1p^{\alpha_1} + a_2p^{\alpha_2} + \cdots + a_sp^{\alpha_s}}{p^i}\right]$$
$$= \begin{cases} \sum_{j=k}^{s} a_jp^{\alpha_j - i}, & \text{if } \alpha_{k-1} < i \leq \alpha_k \\ 0, & \text{if } i > \alpha_s. \end{cases}$$

So from this formula we have

$$\alpha(n) \equiv \sum_{i=1}^{+\infty} \left[\frac{n}{p^i}\right] = \sum_{i=1}^{+\infty} \left[\frac{a_1p^{\alpha_1} + a_2p^{\alpha_2} + \cdots + a_sp^{\alpha_s}}{p^i}\right]$$
$$= \sum_{j=1}^{s} \sum_{k=1}^{\alpha_j} a_jp^{\alpha_j - k} = \sum_{j=1}^{s} a_j\left(1 + p + p^2 + \cdots + p^{\alpha_j - 1}\right)$$
$$= \sum_{j=1}^{s} a_j \cdot \frac{p^{\alpha_j} - 1}{p-1} = \frac{1}{p-1}\sum_{j=1}^{s}\left(a_jp^{\alpha_j} - a_j\right)$$
$$= \frac{1}{p-1}\left(n - a(n,p)\right).$$

This completes the proof of Lemma 1.

Lemma 2. *For any positive integer n with $p|n$, we have the estimate*

$$a(n,p) \leq \frac{p}{\ln p}\ln n.$$

Proof. Let $n = a_1p^{\alpha_1} + a_2p^{\alpha_2} + \cdots + a_sp^{\alpha_s}$ with $\alpha_s > \alpha_{s-1} > \cdots > \alpha_1 \geq 1$, where $1 \leq a_i \leq p-1$, $i = 1, 2, \cdots, s$. Then from the definition of $a(n,p)$ we have

$$a(n,p) = \sum_{i=1}^{s} a_i \leq \sum_{i=1}^{s}(p-1) = (p-1)s. \tag{1}$$

On the other hand, using the mathematical induction we can easily get the inequality

$$n = a_1 p^{\alpha_1} + a_2 p^{\alpha_2} + \cdots + a_s p^{\alpha_s} \geq a_s p^s,$$

or

$$s \leq \frac{\ln(n/a_s)}{\ln p} \leq \frac{\ln n}{\ln p}. \tag{2}$$

Combining (1) and (2) we immediately get the estimate

$$a(n,p) \leq \frac{p}{\ln p} \ln n.$$

This proves the Lemma 2.

Now we use Lemma 1 and Lemma 2 to complete the proof of the Theorem. For any fixed prime p and any positive integer n, let $S_p(n) = k = a_1 \cdot p^{\alpha_1} + a_2 \cdot p^{\alpha_2} + \cdots + a_s \cdot p^{\alpha_s}$ with $\alpha_s > \alpha_{s-1} > \cdots > \alpha_1 \geq 0$ under the base p. Then from the definition of $S_p(n)$ we know that $p^n | k!$ and $p^n \nmid (k-1)!$, so that $\alpha_1 \geq 1$. Note that the factorization of $k!$ into prime powers is

$$k! = \prod_{q \leq k} q^{\alpha_q(k)},$$

where $\prod_{q \leq k}$ denotes the product over all prime $\leq k$, and $\alpha_q(k) = \sum_{i=1}^{+\infty} \left[\frac{k}{q^i}\right]$. From Lemma 1 we immediately get the inequality

$$\alpha_p(k) - \alpha_1 < n \leq \alpha_p(k)$$

or

$$\frac{1}{p-1}(k - a(k,p)) - \alpha_1 < n \leq \frac{1}{p-1}(k - a(k,p)).$$

i.e.

$$(p-1)n + a(k,p) \leq k \leq (p-1)n + a(k,p) + (p-1)(\alpha_1 - 1).$$

Combining this inequality and Lemma 2 we obtain the asymptotic formula

$$k = (p-1)n + O\left(\frac{p}{\ln p} \ln k\right) = (p-1)n + O\left(\frac{p}{\ln p} \ln n\right).$$

This completes the proof of the Theorem.

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ON A EQUATION OF SMARANDACHE AND ITS INTEGER SOLUTIONS*

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ABSTRACT. Let Q denotes the set of all rational numbers, $a \in Q\setminus\{-1,0,1\}$. The main purpose of this paper is to prove that the equation

$$x \cdot a^{\frac{1}{x}} + \frac{1}{x} \cdot a^x = 2a$$

has one and only one integer solution $x = 1$. This solved a problem of Smarandache in book [1].

1. INTRODUCTION

Let Q denotes the set of all rational numbers, $a \in Q\setminus\{-1,0,1\}$. In problem 50 of book [1], Professor F.Smarandache ask us to solve the equation

$$x \cdot a^{\frac{1}{x}} + \frac{1}{x} \cdot a^x = 2a. \tag{1}$$

About this problem, it appears that no one had studied it yet, at least, we have not seen such a result before. The problem is interesting because it can help us to understand some new indefinite equations. In this paper, we use elementary method and analysis method to study the equation (1), and prove the following conclusion:

Theorem. *For all $a \in Q\setminus\{-1,0,1\}$, the equation*

$$x \cdot a^{\frac{1}{x}} + \frac{1}{x} \cdot a^x = 2a$$

has one and only one integer solution $x = 1$.

Key words and phrases. F.Smarandache equation; Integer solution; One and only one solution.
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2. Proof of the Theorem

In this section, we use elementary methods and the Rolle's Theorem in mathematical analysis to complete the proof of the Theorem. First we prove that the Theorem holds for $a > 1$. In fact in this case, let x is an integer solution of (1), we must have $x > 0$. Then using the inequality $|u| + |v| \geq 2\sqrt{|u| \cdot |v|}$ we have

$$x \cdot a^{\frac{1}{x}} + \frac{1}{x} \cdot a^x \geq 2 \cdot \sqrt{x \cdot a^{\frac{1}{x}} \cdot \frac{1}{x} \cdot a^x} = 2 \cdot a^{\frac{x+\frac{1}{x}}{2}} \geq 2 \cdot a,$$

and the equality holds if and only if $x = 1$. This proved that for $a > 1$, the equation (1) has one and only one integer solution $x = 1$.

Now we consider $0 < a < 1$. Let x_0 is any integer solution of (1), then from equation (1) we know that $x_0 > 0$. To prove $x_0 = 1$, we suppose $x_0 \neq 1$, let $0 < x_0 < 1$ (the proof for case $x_0 > 1$ is the same as for $0 < x_0 < 1$), then $\frac{1}{x_0} > 1$, we define the function $f(x)$ as follows:

$$f(x) = x \cdot a^{\frac{1}{x}} + \frac{1}{x} \cdot a^x - 2a$$

It is clear that $f(x)$ is a continuous function in the closed interval $\left[x_0, \frac{1}{x_0}\right]$, and a derivable function in the open interval $\left(x_0, \frac{1}{x_0}\right)$, and more $f(x_0) = f(\frac{1}{x_0}) = f(1) = 0$. So from the Rolle's Theorem in mathematical analysis we know that $f'(x)$ must have two zero points in the open interval $\left(x_0, \frac{1}{x_0}\right)$, and $f''(x)$ must have one zero point in the same open interval. But from the definition of $f(x)$ we have

$$f'(x) = a^{\frac{1}{x}} - \frac{1}{x} \cdot a^{\frac{1}{x}} \cdot \ln a - \frac{1}{x^2} \cdot a^x + \frac{1}{x} \cdot a^x \cdot \ln a$$

and

$$\begin{aligned} f''(x) &= \frac{1}{x^3} \cdot a^{\frac{1}{x}} \cdot \ln^2 a + \frac{2}{x^3} \cdot a^x - \frac{1}{x^2} \cdot a^x \cdot \ln a - \frac{1}{x^2} \cdot a^x \cdot \ln a + \frac{1}{x} \cdot a^x \cdot \ln^2 a \\ &= \frac{1}{x^3} \cdot a^{\frac{1}{x}} \cdot \ln^2 a + \frac{2}{x^3} \cdot a^x + \frac{2}{x^2} \cdot a^x \cdot \ln \frac{1}{a} + \frac{1}{x} \cdot a^x \cdot \ln^2 a \\ &> 0, \quad x \in \left(x_0, \frac{1}{x_0}\right), \end{aligned}$$

where we have used $0 < a < 1$ and $\ln \frac{1}{a} > 0$. This contradiction with that $f''(x)$ must have one zero point in the open interval $\left(x_0, \frac{1}{x_0}\right)$. This proved that the Theorem holds for $0 < a < 1$.

If $a < 0$ and $a \neq -1$, and equation (1) has an integer solution x, then $|x|$ must be an odd number, because negative number has no real square root. So in this

case, the equation (1) become the following equation:

$$2|a| = -2a = -x \cdot a^{\frac{1}{x}} - \frac{1}{x} \cdot a^x = -x \cdot (-1)^{\frac{1}{x}} \cdot |a|^{\frac{1}{x}} - \frac{1}{x} \cdot (-1)^x \cdot |a|^x$$
$$= x \cdot |a|^{\frac{1}{x}} + \frac{1}{x} \cdot |a|^x.$$

Then from the above conclusion we know that the Theorem is also holds. This completes the proof of the Theorem.

Note. In fact from the process of the proof of the Theorem we can easily find that we have proved a more general conclusion:

Theorem. *Let R denotes the set of all real numbers. For any $a \in R \setminus \{-1, 0, 1\}$, the equation*

$$x \cdot a^{\frac{1}{x}} + \frac{1}{x} \cdot a^x = 2a$$

has one and only one integer solution $x = 1$; It has one and only one real number solution $x = 1$, if $a > 0$.

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A NEW SEQUENCE RELATED SMARANDACHE SEQUENCES AND ITS MEAN VALUE FORMULA*

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ABSTRACT. Let n be any positive integer, $a(n)$ denotes the product of all non-zero digits in base 10. For natural $x \geq 2$ and arbitrary fixed exponent $m \in N$, let $A_m(x) = \sum_{n<x} a^m(n)$. The main purpose of this paper is to give two exact calculating formulas for $A_1(x)$ and $A_2(x)$.

1. INTRODUCTION

For any positive integer n, let $b(n)$ denotes the product of base 10 digits of n. For example, $b(1) = 1$, b(2)=2,$\cdots$, $b(10) = 0$, $b(11) = 1$, $\cdots\cdots$. In problem 22 of book [1], Professor F.Smaradache ask us to study the properties of sequence $\{b(n)\}$. About this problem, it appears that no one had studied it yet, at least, we have not seen such a paper before. The problem is interesting because it can help us to find some new distribution properties of the base 10 digits. In this paper, we consider another sequence $\{a(n)\}$, which related to Smarandache sequences. Let $a(n)$ denotes the product of all non-zero digits in base 10 of n. For example, $a(1) = 1$, $a(2) = 2$, $a(12) = 2$, $\cdots$, $a(28) = 16$, $a(1023) = 6, \cdots\cdots$. For natural number $x \geq 2$ and arbitrary fixed exponent $m \in N$, let

$$A_m(x) = \sum_{n<x} a^m(n). \tag{1}$$

The main purpose of this paper is to study the calculating problem of $A_m(x)$, and use elementary methods to deduce two exact calculating formulas for $A_1(x)$ and $A_2(x)$. That is, we shall prove the following:

Theorem. *For any positive integer x, let $x = a_1 10^{k_1} + a_2 10^{k_2} + \cdots + a_s 10^{k_s}$ with $k_1 > k_2 > \cdots > k_s \geq 0$ and $1 \leq a_i \leq 9$, $i = 2, 3, \cdots, s$. Then we have the calculating formulas*

$$A_1(x) = \frac{a_1 a_2 \cdots a_s}{2} \sum_{i=1}^{s} \frac{a_i^2 - a_i + 2}{\prod_{j=i}^{s} a_j} \left(45 + \left[\frac{1}{k_i + 1}\right]\right) \cdot 46^{k_i - 1};$$

Key words and phrases. F.Smarandache sequence; The base 10 digits; Calculating formula.
* This work is supported by the N.S.F. and the P.S.F. of P.R.China.

$$A_2(x)=\frac{a_1^2a_2^2\cdots a_s^2}{6}\sum_{i=1}^{s}\frac{2a_i^3-3a_i^2+a_i+6}{\prod\limits_{j=i}^{s}a_j^2}\left(285+\left[\frac{1}{k_i+1}\right]\right)\cdot 286^{k_i-1},$$

where $[x]$ *denotes the greatest integer not exceeding* x.

For general integer $m\geq 3$, using our methods we can also give an exact calculating formula for $A_m(x)$. That is, we have the calculating formula

$$A_m(x)=a_1^ma_2^m\cdots a_s^m\sum_{i=1}^{s}\frac{1+B_m(a_i)}{\prod\limits_{j=i}^{s}a_j^m}\left(\left[\frac{1}{k_i+1}\right]+B_m(10)\right)\cdot(1+B_m(10))^{k_i-1},$$

where a_i as the definition as in the above Theorem, and $B_m(N)=\sum\limits_{1\leq n<N}n^m$.

2. Proof of the Theorem

In this section, we complete the proof of the Theorem. First we need following two simple Lemmas.

Lemma 1. *For any integer* $k\geq 1$ *and* $1\leq c\leq 9$, *we have the identities*

a) $A_1(10^k)=45\cdot 46^{k-1}$;

b) $A_1(c\cdot 10^k)=45\cdot\left(1+\frac{(c-1)c}{2}\right)\cdot 46^{k-1}$.

Proof. We first prove a) of Lemma 1 by induction. For $k=1$, we have $A_1(10^1)=A_1(10)=1+2+\cdots+9=45$. So that the identity

$$A_1(10^k)=\sum_{n<10^k}a(n)=45\cdot 46^{k-1} \tag{2}$$

holds for $k=1$. Assume (2) is true for $k=m\geq 1$. Then by the inductive assumption we have

$$\begin{aligned}
A_1(10^{m+1}) &= \sum_{n<9\cdot 10^m}a(n)+\sum_{9\cdot 10^m\leq n<10^{m+1}}a(n)\\
&= A_1(9\cdot 10^m)+\sum_{0\leq n<10^m}a(n+9\cdot 10^m)\\
&= A_1(9\cdot 10^m)+9\cdot\sum_{0\leq n<10^m}a(n)\\
&= A_1(9\cdot 10^m)+9\cdot\sum_{n<10^m}a(n)\\
&= A_1(9\cdot 10^m)+9\cdot A_1(10^m)\\
&= A_1(8\cdot 10^m)+9\cdot A_1(10^m)+8\cdot A_1(10^m)\\
&= \cdots\cdots\cdots\\
&= (1+1+2+3+4+5+6+7+8+9)\cdot A_1(10^m)\\
&= 46\cdot A_1(10^m)\\
&= 45\cdot 46^m.
\end{aligned}$$

That is, (2) is true for $k = m+1$. This proves the first part of Lemma 1.

The second part b) follows from a) of Lemma 1 and the recurrence formula

$$\begin{aligned} A_1(c\cdot 10^k) &= \sum_{n<(c-1)\cdot 10^k} a(n) + \sum_{(c-1)\cdot 10^k \le n < c\cdot 10^k} a(n) \\ &= \sum_{n<(c-1)\cdot 10^k} a(n) + \sum_{0\le n<10^k} a(n+(c-1)\cdot 10^k) \\ &= \sum_{n<(c-1)\cdot 10^k} a(n) + (c-1)\cdot \sum_{n<10^k} a(n) \\ &= A_1((c-1)\cdot 10^k) + (c-1)\cdot A_1(10^k). \end{aligned}$$

This completes the proof of Lemma 1.

Lemma 2. *For any integer $k \ge 1$ and $1 \le c \le 9$, we have the identities*

c) $A_2(10^k) = 285\cdot 286^{k-1}$;

d) $A_2(a\cdot 10^k) = 285\cdot\left[1+\dfrac{(a-1)a(2a-1)}{6}\right]\cdot 286^{k-1}$.

Proof. Note that $A_2(10) = 285$. The Lemma 2 can be deduced by Lemma 1, induction and the recurrence formula

$$\begin{aligned} A_2(10^{k+1}) &= \sum_{n<9\cdot 10^k} a^2(n) + \sum_{9\cdot 10^k \le n < 10^{k+1}} a^2(n) \\ &= \sum_{n<9\cdot 10^k} a^2(n) + \sum_{0\le n<10^k} a^2(n+9\cdot 10^k) \\ &= \sum_{n<9\cdot 10^k} a^2(n) + 9^2\cdot \sum_{0\le n<10^k} a^2(n) \\ &= A_2(9\cdot 10^k) + 9^2\cdot A_2(10^k) \\ &= \cdots\cdots\cdots \\ &= (1+1^2+2^2+\cdots+9^2)\cdot A_2(10^k) \\ &= 286\cdot A_2(10^k). \end{aligned}$$

This completes the proof of Lemma 2.

Now we use Lemma 1 and Lemma 2 to complete the proof of the Theorem. For any positive integer x, let $x = a_1\cdot 10^{k_1} + a_2\cdot 10^{k_2} + \cdots + a_s\cdot 10^{k_s}$ with $k_1 > k_2 > \cdots > k_s \ge 0$ under the base 10. Then applying Lemma 1 repeatedly we have

$$\begin{aligned} A_1(x) &= \sum_{n<a_1\cdot 10^{k_1}} a(n) + \sum_{a_1\cdot 10^{k_1}\le n<x} a(n) \\ &= A_1(a_1\cdot 10^{k_1}) + \sum_{0\le n<x-a_1\cdot 10^{k_1}} a(n+a_1\cdot 10^{k_1}) \\ &= A_1(a_1\cdot 10^{k_1}) + a_1\cdot \sum_{0\le n<x-a_1\cdot 10^{k_1}} a(n) \end{aligned}$$

$$= A_1(a_1 \cdot 10^{k_1}) + a_1 \cdot A_1(x - a_1 \cdot 10^{k_1})$$
$$= A_1(a_1 \cdot 10^{k_1}) + a_1 \cdot A_1(a_2 \cdot 10^{k_2}) + a_1 a_2 \cdot A_1(x - a_1 \cdot 10^{k_1} - a_2 \cdot 10^{k_2})$$
$$= \cdots\cdots\cdots$$
$$= \sum_{i=1}^{s} \frac{a_1 a_2 \cdots a_s}{a_i a_{i+1} \cdots a_s} A_1(a_i \cdot 10^{k_i})$$
$$= a_1 a_2 \cdots a_s \sum_{i=1}^{s} \frac{(1 + \frac{(a_i - 1)a_i}{2})}{\prod_{j=i}^{s} a_j} \left(45 + \left[\frac{1}{k_i + 1}\right]\right) 46^{k_i - 1}.$$

This proves the first part of the Theorem.

Applying Lemma 2 and the first part of the Theorem repeatedly we have

$$A_2(x) = \sum_{n < a_1 \cdot 10^{k_1}} a^2(n) + \sum_{a_1 \cdot 10^{k_1} \le n < x} a^2(n)$$
$$= A_2(a_1 \cdot 10^{k_1}) + \sum_{0 \le n < x - a_1 \cdot 10^{k_1}} a^2(n + a_1 \cdot 10^{k_1})$$
$$= A_2(a_1 \cdot 10^{k_1}) + a_1^2 \cdot \sum_{0 \le n < x - a_1 \cdot 10^{k_1}} a^2(n)$$
$$= A_2(a_1 : 10^{k_1}) + a_1^2 \cdot A_2(x - a_1 \cdot 10^{k_1})$$
$$= A_2(a_1 \cdot 10^{k_1}) + a_1^2 \cdot A_2(a_2 \cdot 10^{k_2}) + a_1^2 a_2^2 \cdot A_2(x - a_1 \cdot 10^{k_1} - a_2 \cdot 10^{k_2})$$
$$= \cdots\cdots\cdots$$
$$= \sum_{i=1}^{s} \frac{a_1^2 a_2^2 \cdots a_s^2}{\prod_{j=i}^{s} a_j^2} A_2(a_i \cdot 10^{k_i})$$
$$= \frac{a_1^2 a_2^2 \cdots a_s^2}{6} \sum_{i=1}^{s} \frac{2a_i^3 - 3a_i^2 + a_i + 6}{\prod_{j=i}^{s} a_j^2} \left(285 + \left[\frac{1}{k_i + 1}\right]\right) \cdot 286^{k_i - 1}.$$

This completes the proof of the second part of the Theorem.

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Five Properties of the Smarandache Double Factorial Function

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Abstract

In this paper some properties of the Smarandache double factorial function have been analyzed.

In [1], [2], [3] and [4] the Smarandache double factorial Sdf(n) function is defined as the smallest number such that Sdf(n)!! is divisible by n, where the double factorial by definition is given by [6]:

m!! = 1x3x5x...m, if m is odd;
m!! = 2x4x6x...m, if m is even.

In [2] several properties of that function have been analyzed. In this paper five new properties are reported.

1. $Sdf(p^{k+2}) = p^2$ *where* $p = 2 \cdot k + 1$ is any prime and k any integer

Let's consider the prime $p = 2k + 1$. Then:

$1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots\ldots\ldots \cdot p \cdot \ldots\ldots\ldots \cdot 3p \cdot \ldots\ldots\ldots \cdot 5p \cdot \ldots\ldots\ldots \cdot p^2 = m \cdot p^{k+2}$ where m is any integer.

This because the number of terms multiples of p up to p^2 are k+1 and the last term contains two times p.
Then p^2 is the least value such that $1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot \ldots\ldots \cdot p^2$ is divisible by p^{k+2}.

2. $Sdf(p^2) = 3 \cdot p$ where p is any odd prime.

In fact for any odd p we have:

$1\cdot 3\cdot 5\cdot 7\cdot .....\cdot p\cdot .....\cdot 3p = m\cdot p^2$ where m is any integer.

3. $Sdf\left(k\cdot\left(\frac{10^n-1}{9}\right)\right) = Sdf\left(\frac{10^n-1}{9}\right)$ where n is any integer >1 and k=3,5,7,9

Let's suppose that $Sdf\left(\frac{10^n-1}{9}\right) = m$ then:

$1\cdot 3\cdot 5\cdot 7\cdot .....\cdot m = a\cdot\left(\frac{10^n-1}{9}\right)$ where a is any integer. But in the previous multiplication there are factors multiple of 3,5,7 and 9 and then:

$1\cdot 3\cdot 5\cdot 7\cdot .....\cdot m = a'\cdot k\cdot\left(\frac{10^n-1}{9}\right)$ where a' is any integer and k=3,5,7,9. Then:

$$Sdf\left(k\cdot\left(\frac{10^n-1}{9}\right)\right) = m = Sdf\left(\frac{10^n-1}{9}\right)$$

4. $Sdf\left(k\cdot\left(\frac{10^n-1}{9}\right)\right) = Sdf\left(2\cdot\left(\frac{10^n-1}{9}\right)\right)$ where n is any integer >1 and k=2,4,6,8

Let's suppose that $Sdf\left(2\cdot\left(\frac{10^n-1}{9}\right)\right) = m$ then:

$2\cdot 4\cdot 6\cdot 8\cdot .....\cdot m = a\cdot 2\cdot\left(\frac{10^n-1}{9}\right)$ where a is any integer. But in the previous

multiplication there are factors multiple of 4, 6 and 8 and then:

$2\cdot 4\cdot 6\cdot 8\cdot .....\cdot m = a'\cdot 2\cdot k\cdot\left(\frac{10^n-1}{9}\right)$ where a' is any integer and k=4,6,8.

Then:

$$Sdf\left(k\cdot\left(\frac{10^n-1}{9}\right)\right) = m = Sdf\left(2\cdot\left(\frac{10^n-1}{9}\right)\right)$$

5. $Sdf(p^m) = (2\cdot m-1)\cdot p$ *for* $p \geq (2m-1)$. Here m is any integer and p any odd prime.

This is a generalization of property number 2 reported above.

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ON A PROBLEM OF F. SMARANDACHE*

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ABSTRACT. Let $d_s(n)$ denotes the sum of the base 10 digits of $n \in N$. For natural $x \geq 2$ and arbitrary fixed exponent $m \in N$, let $A_m(x) = \sum_{n<x} d_s^m(n)$. The main purpose of this paper is to give two exact calculating formulas for $A_1(x)$ and $A_2(x)$.

1. INTRODUCTION

For any positive integer n, let $d_s(n)$ denotes the sum of the base 10 digits of n. For example, $d_s(0) = 0$, $d_s(1) = 1$, $d_s(2) = 2, \cdots$, $d_s(11) = 2$, $d_s(12) = 3, \cdots\cdots$. In problem 21 of book [1], Professor F.Smaradache ask us to study the properties of sequence $\{d_s(n)\}$. For natural number $x \geq 2$ and arbitrary fixed exponent $m \in N$, let

$$A_m(x) = \sum_{n<x} d_s^m(n). \tag{1}$$

The main purpose of this paper is to study the calculating problem of $A_m(x)$, and use elementary methods to deduce two exact calculating formulas for $A_1(x)$ and $A_2(x)$. That is, we shall prove the following:

Theorem. *For any positive integer x, let $x = a_1 10^{k_1} + a_2 10^{k_2} + \cdots + a_s 10^{k_s}$ with $k_1 > k_2 > \cdots > k_s \geq 0$ and $1 \leq a_i \leq 9$, $i = 2, 3, \cdots, s$. Then we have the calculating formulas*

$$A_1(x) = \sum_{i=1}^{s} a_i \cdot \left(\frac{9}{2} k_i + \sum_{j=1}^{i} a_j - \frac{a_i + 1}{2} \right) \cdot 10^{k_i};$$

$$A_2(x) = \sum_{i=1}^{s} a_i \cdot \left[\frac{k_i(81k_i + 33)}{4} + \frac{9k_i}{2}(a_i - 1) + \sum_{j=1}^{i} a_j^2 - \frac{(4a_i - 1)(a_i + 1)}{6} \right] \cdot 10^{k_i}$$
$$+ \sum_{i=2}^{s} a_i \cdot \left[(9k_i - a_i - 1)10^{k_i} + 2 \sum_{j=i}^{s} a_j 10^{k_j} \right] \cdot \left(\sum_{j=1}^{i-1} a_j \right).$$

For general integer $m \geq 3$, using our methods we can also give an exact calculating formula for $A_m(x)$. But in these cases, the computations are more complex.

Key words and phrases. F.Smarandache problem; Sum of base 10 digits; Calculating formula.
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2. Proof of the Theorem

In this section, we complete the proof of the Theorem. First we need following two simple Lemmas.

Lemma 1. *For any integer $k \geq 0$, we have the identities*

a) $A_1(10^k) = \frac{9}{2} \cdot k \cdot 10^k$;

b) $A_1(a \cdot 10^k) = \left(\frac{9}{2}k + \frac{a-1}{2}\right) \cdot a \cdot 10^k, \quad 1 \leq a \leq 9.$

Proof. We first prove a) of Lemma 1 by induction. For $k = 0$ and 1, we have $A_1(10^0) = A_1(1) = 0$, $A_1(10^1) = A_1(10) = 45$. So that the identity

$$A_1(10^k) = \sum_{n<10^k} d_s(n) = \frac{9}{2} \cdot k \cdot 10^k \tag{2}$$

holds for $k = 0$ and 1. Assume (2) is true for $k = m - 1$. Then by the inductive assumption we have

$$\begin{aligned}
A_1(10^m) &= \sum_{n<9\cdot 10^{m-1}} d_s(n) + \sum_{9\cdot 10^{m-1}\leq n<10^m} d_s(n) \\
&= A_1(9 \cdot 10^{m-1}) + \sum_{0\leq n<10^{m-1}} d_s(n + 9 \cdot 10^{m-1}) \\
&= A_1(9 \cdot 10^{m-1}) + \sum_{0\leq n<10^{m-1}} (d_s(n) + 9) \\
&= A_1(9 \cdot 10^{m-1}) + 9 \cdot 10^{m-1} + \sum_{n<10^{m-1}} d_s(n) \\
&= A_1(9 \cdot 10^{m-1}) + 9 \cdot 10^{m-1} + A_1(10^{m-1}) \\
&= A_1(8 \cdot 10^{m-1}) + (8+9) \cdot 10^{m-1} + 2A_1(10^{m-1}) \\
&= \cdots\cdots\cdots \\
&= (1+2+3+4+5+6+7+8+9) \cdot 10^{m-1} + 10A_1(10^{m-1}) \\
&= \frac{9}{2} \cdot 10^m + 10 \cdot \frac{9}{2} \cdot (m-1) \cdot 10^{m-1} \\
&= \frac{9}{2} \cdot m \cdot 10^m.
\end{aligned}$$

That is, (2) is true for $k = m$. This proves the first part of Lemma 1.

The second part b) follows from a) of Lemma 1 and the recurrence formula

$$\begin{aligned}
A_1(a \cdot 10^k) &= \sum_{n<(a-1)\cdot 10^k} d_s(n) + \sum_{(a-1)\cdot 10^k\leq n<a\cdot 10^k} d_s(n) \\
&= \sum_{n<(a-1)\cdot 10^k} d_s(n) + \sum_{0\leq n<10^k} d_s(n + (a-1) \cdot 10^k) \\
&= \sum_{n<(a-1)\cdot 10^k} a(n) + (a-1) \cdot 10^k + \sum_{n<10^k} d_s(n) \\
&= A_1((a-1) \cdot 10^k) + (a-1) \cdot 10^k + A_1(10^k).
\end{aligned}$$

This completes the proof of Lemma 1.

Lemma 2. *For any integer $k \geq 0$ and $1 \leq a \leq 9$, we have the identities*

c) $A_2(10^k) = \dfrac{81k+33}{4} \cdot k \cdot 10^k$;

d) $A_2(a \cdot 10^k) = \left[\dfrac{k(81k+33)}{4} + \dfrac{9k}{2}(a-1) + \dfrac{(a-1)(2a-1)}{6}\right] \cdot a \cdot 10^k$.

Proof. These results can be deduced by Lemma 1, induction and the recurrence formula

$$\begin{aligned}
A_2(10^{k+1}) &= \sum_{n<9\cdot 10^k} d_s^2(n) + \sum_{9\cdot 10^k \leq n < 10^{k+1}} d_s^2(n) \\
&= \sum_{n<9\cdot 10^k} d_s^2(n) + \sum_{0\leq n<10^k} d_s^2(n+9\cdot 10^k) \\
&= \sum_{n<9\cdot 10^k} d_s^2(n) + \sum_{0\leq n<10^k} (d_s(n)+9)^2 \\
&= A_2(9\cdot 10^k) + 9^2\cdot 10^k + 18A_1(10^k) + A_2(10^k) \\
&= \cdots\cdots\cdots \\
&= 10A_2(10^k) + (1^2+2^2+\cdots+9^2)\cdot 10^k + 2\cdot(1+2+\cdots+9)A_1(10^k) \\
&= 10A_2(10^k) + \frac{57}{2}\cdot 10^{k+1} + 90\cdot\frac{9}{2}\cdot k\cdot 10^k \\
&= 10A_2(10^k) + \frac{57}{2}\cdot 10^{k+1} + \frac{81}{2}\cdot k\cdot 10^{k+1}.
\end{aligned}$$

This completes the proof of Lemma 2.

Now we use Lemma 1 and Lemma 2 to complete the proof of the Theorem. For any positive integer x, let $x = a_1\cdot 10^{k_1} + a_2\cdot 10^{k_2} + \cdots + a_s\cdot 10^{k_s}$ with $k_1 > k_2 > \cdots > k_s \geq 0$ under the base 10. Then applying Lemma 1 repeatedly we have

$$\begin{aligned}
A_1(x) &= \sum_{n<a_1\cdot 10^{k_1}} d_s(n) + \sum_{a_1\cdot 10^{k_1}\leq n<x} d_s(n) \\
&= A_1(a_1\cdot 10^{k_1}) + \sum_{0\leq n<x-a_1\cdot 10^{k_1}} d_s(n+a_1\cdot 10^{k_1}) \\
&= A_1(a_1\cdot 10^{k_1}) + \sum_{0\leq n<x-a_1\cdot 10^{k_1}} (d_s(n)+a_1) \\
&= A_1(a_1\cdot 10^{k_1}) + a_1(x-a_1\cdot 10^{k_1}) + \sum_{0\leq n<x-a_1\cdot 10^{k_1}} d_s(n) \\
&= A_1(a_1\cdot 10^{k_1}) + a_1(x-a_1\cdot 10^{k_1}) + A_1(x-a_1\cdot 10^{k_1}) \\
&= A_1(a_1\cdot 10^{k_1}) + A_1(a_2\cdot 10^{k_2}) + a_1(x-a_1\cdot 10^{k_1}) \\
&\quad + a_2(x-a_1\cdot 10^{k_1}-a_2\cdot 10^{k_2}) + A_1(x-a_1\cdot 10^{k_1}-a_2\cdot 10^{k_2}) \\
&= \cdots\cdots\cdots
\end{aligned}$$

$$= \sum_{i=1}^{s} A_1(a_i \cdot 10^{k_i}) + \sum_{i=1}^{s} a_i(x - \sum_{j=1}^{i} a_j \cdot 10^{k_j})$$

$$= \sum_{i=1}^{s} \left(\frac{9}{2} \cdot k_i + \frac{a_i - 1}{2}\right) \cdot a_i \cdot 10^{k_i} + \sum_{i=2}^{s} a_i \cdot 10^{k_i} \left(\sum_{j=1}^{i-1} a_j\right)$$

$$= \sum_{i=1}^{s} \left(\frac{9}{2} k_i + \sum_{j=1}^{i} a_j - \frac{a_i + 1}{2}\right) \cdot a_i \cdot 10^{k_i}.$$

This proves the first part of the Theorem.

Applying Lemma 2 and the first part of the Theorem repeatedly we have

$$A_2(x) = \sum_{n < a_1 \cdot 10^{k_1}} d_s^2(n) + \sum_{a_1 \cdot 10^{k_1} \le n < x} d_s^2(n)$$

$$= A_2(a_1 \cdot 10^{k_1}) + \sum_{0 \le n < x - a_1 \cdot 10^{k_1}} d_s^2(n + a_1 \cdot 10^{k_1})$$

$$= A_2(a_1 \cdot 10^{k_1}) + \sum_{0 \le n < x - a_1 \cdot 10^{k_1}} (d_s(n) + a_1)^2$$

$$= A_2(a_1 \cdot 10^{k_1}) + \sum_{0 \le n < x - a_1 \cdot 10^{k_1}} (d_s^2(n) + 2a_1 \cdot d_s(n) + a_1^2)$$

$$= A_2(a_1 \cdot 10^{k_1}) + a_1^2 \cdot (x - a_1 \cdot 10^{k_1}) + 2a_1 A_1(x - a_1 \cdot 10^{k_1}) + A_2(x - a_1 \cdot 10^{k_1})$$

$$= \cdots\cdots\cdots$$

$$= \sum_{i=1}^{s} A_2(a_i \cdot 10^{k_i}) + \sum_{i=1}^{s} a_i^2(x - \sum_{j=1}^{i} a_j \cdot 10^{k_j}) + \sum_{i=1}^{s} 2a_i A_1(x - \sum_{j=1}^{i} a_j \cdot 10^{k_j})$$

$$= \sum_{i=1}^{s} \left[\frac{k_i(81k_i + 33)}{4} + \frac{9k_i}{2}(a_i - 1) + \frac{(a_i - 1)(2a_i - 1)}{6}\right] \cdot a_i \cdot 10^{k_i}$$

$$+ \sum_{i=2}^{s} a_i \cdot 10^{k_i} \cdot \left(\sum_{j=1}^{i-1} a_j^2\right) + \sum_{i=2}^{s} (9k_i + a_i - 1) \cdot a_i \cdot 10^{k_i} \cdot \left(\sum_{j=1}^{i-1} a_j\right)$$

$$+ 2\sum_{i=2}^{s} \left(\sum_{j=1}^{i-1} a_j\right) \cdot a_i \cdot \left(x - \sum_{j=1}^{i} a_j \cdot 10^{k_j}\right)$$

$$= \sum_{i=1}^{s} \left[\frac{k_i(81k_i + 33)}{4} + \frac{9k_i}{2}(a_i - 1) + \sum_{j=1}^{i} a_j^2 - \frac{(4a_i - 1)(a_i + 1)}{6}\right] \cdot a_i \cdot 10^{k_i}$$

$$+ \sum_{i=2}^{s} a_i \cdot \left[(9k_i - a_i - 1)10^{k_i} + 2\sum_{j=i}^{s} a_j 10^{k_j}\right] \cdot \left(\sum_{j=1}^{i-1} a_j\right).$$

This completes the proof of the second part of the Theorem.

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On the Smarandache Lucas base and related counting function[1]

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1. Introduction and results

As usual, the Lucas sequence $\{L_n\}$ and the Fibonacci sequence $\{F_n\}$ $(n = 0, 1, 2, \ldots,)$ are defined by the second-order linear recurrence sequences

$$L_{n+2} = L_{n+1} + L_n \qquad \text{and} \qquad F_{n+2} = F_{n+1} + F_n$$

for $n \geq 0$, $L_0 = 2$, $L_1 = 1$, $F_0 = 0$ and $F_1 = 1$. These sequences play a very important role in the studies of the theory and application of mathematics. Therefore, the various properties of L_n and F_n were investigated by many authors. For example, R. L. Duncan [1] and L. Kuipers [2] proved that $(\log F_n)$ is uniformly distributed mod 1. H.London and R.Finkelstein [3] studied the Fibonacci and Lucas numbers which are perfect powers. The author [4] obtained some identities involving the Fibonacci numbers.

In this paper, we introduce a new counting function $a(m)$ related to the Lucas numbers, then use elementary methods to give an exact calculating formula for its mean value. First we consider the Smarandache's generalized base, Professor F.Smarandach defined over the set of natural numbers the following infinite generalized base: $1 = g_0 < g_1 < \cdots < g_k < \cdots$. He proved that every positive integer N may be uniquely written in the Smarandache Generalized Base as:

$$N = \sum_{i=0}^{n} a_i g_i, \quad \text{with} \quad 0 \leq a_i \leq \left[\frac{g_{i+1} - 1}{g_i}\right]$$

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(integer part) for $i = 0, 1, \cdots, n$, and of course $a_n \geq 1$, in the following way: if $g_n \leq N < g_{n+1}$, then $N = g_n + r_1$; if $g_m \leq r_1 < g_{m+1}$, then $r_1 = g_m + r_2$, $m < n$; and so on untill one obtains a rest $r_j = 0$.

This base is important for partitions. If we take the g_i as the Lucas sequence, then we can get a particular base, for convenience, we refer to it as a Smarandache Lucas base. Then any positive integer m may be uniquely written in the Smarandache Lucas base as:

$$m = \sum_{i=1}^{n} a_i L_i, \quad \text{with all } a_i = 0 \text{ or } 1, \tag{1}$$

That is, any positive integer may be written as a sum of Lucas numbers. Now for an integer $m = \sum_{i=1}^{n} a_i L_i$, we define the counting function $a(m) = a_1 + a_2 + \cdots + a_n$. The main purpose of this paper is to study the distribution properties of $a(m)$, and present a calculating formula for the mean value

$$A_r(N) = \sum_{n<N} a^r(n), \ r = 1, \ 2. \tag{2}$$

That is, we prove the following two main conclusions:

Theorem 1. *For any positive integer k, we have the calculating formulae*

$$A_1(L_k) = \sum_{n<L_k} a(n) = kF_{k-1}$$

and

$$A_2(L_k) = \frac{1}{5}\left[(k-1)(k-2)L_{k-2} + 5(k-1)F_{k-2} + 7(k-1)F_{k-3} + 3F_{k-1}\right].$$

Theorem 2. *For any positive integer N, let $N = L_{k_1} + L_{k_2} + \cdots + L_{k_s}$ with $k_1 > k_2 > \cdots > k_s$ under the Smarandache Lucas base. Then we have*

$$A_1(N) = A_1(L_{k_1}) + N - L_{k_1} + A_1(N - L_{k_1})$$

and

$$A_2(N) = A_2(L_{k_1}) + N - L_{k_1} + A_2(N - L_{k_1}) + 2A_1(N - L_{k_1}).$$

Further,

$$A_1(N) = \sum_{i=1}^{s} [k_i F_{k_i-1} + (i-1)L_{k_i}].$$

For any positive integer $r \geq 3$, using our methods we can also give an exact calculating formula for $A_r(L_k)$. But in these cases, the computations are more complex.

2. Proof of the Theorems

In this section, we complete the proof of the Theorems. First we prove Theorem 1 by induction. For $k = 1$, 2, we have $A_1(L_1) = A_1(1) = 0$, $A_1(L_2) = A_1(3) = 2$ and $F_0 = 0$, $2F_1 = 2$. So that the identity

$$A_1(L_k) = \sum_{n<L_k} a(n) = kF_{k-1} \tag{3}$$

holds for $k = 1$ and 2. Assume (3) is true for all $k \leq m-1$. Then by the inductive assumption we have

$$\begin{aligned}
A_1(L_m) &= \sum_{n<L_{m-1}} a(n) + \sum_{L_{m-1}\leq n<L_m} a(n) \\
&= A_1(L_{m-1}) + \sum_{0\leq n<L_{m-2}} a(n+L_{m-1}) \\
&= A_1(L_{m-1}) + \sum_{0\leq n<L_{m-2}} (a(n)+1) \\
&= A_1(L_{m-1}) + L_{m-2} + \sum_{n<L_{m-2}} a(n) \\
&= A_1(L_{m-1}) + A_1(L_{m-2}) + L_{m-2} \\
&= (m-1)F_{m-2} + (m-2)F_{m-3} + L_{m-2} \\
&= m(F_{m-2}+F_{m-3}) - F_{m-2} - 2F_{m-3} + L_{m-2} \\
&= mF_{m-1} - F_{m-1} - F_{m-3} + L_{m-2} \\
&= mF_{m-1},
\end{aligned}$$

where we have used the identity $F_{m-1} + F_{m-3} = L_{m-2}$. That is, (3) is true for $k = m$. This proves the first part of Theorem 1.

Now we prove the second part of Theorem 1. For $k = 1, 2$, note that $1 = F_1 = F_0 + F_{-1}$ or $F_{-1} = 1$, we have $A_2(L_1) = A_2(1) = 0$, $A_2(L_2) = A_2(3) = 2$ and

$$\frac{1}{5}[(k-1)(k-2)L_{k-2} + 5(k-1)F_{k-2} + 7(k-1)F_{k-3} + 3F_{k-1}] = \begin{cases} 0, & \text{if } k = 1; \\ 2, & \text{if } k = 2. \end{cases}$$

So that the identity

$$A_2(L_k) = \frac{1}{5}[(k-1)(k-2)L_{k-2} + 5(k-1)F_{k-2} + 7(k-1)F_{k-3} + 3F_{k-1}] \quad (4)$$

holds for $k = 1, 2$. Assume (4) is true for all $k \leq m-1$. Then by the inductive assumption, the first part of Theorem 1 and note that $L_{m-1} + 2L_{m-2} = 5F_{m-1}$ and $F_{m-1} + 2F_{m-2} = L_{m-1}$, we have

$$\begin{aligned}
A_2(L_m) &= \sum_{n<L_{m-1}} a^2(n) + \sum_{L_{m-1}\leq n<L_m} a^2(n) \\
&= A_2(L_{m-1}) + \sum_{0\leq n<L_{m-2}} a^2(n + L_{m-1}) \\
&= A_2(L_{m-1}) + \sum_{0\leq n<L_{m-2}} (a(n)+1)^2 \\
&= A_2(L_{m-1}) + \sum_{0\leq n<L_{m-2}} (a^2(n) + 2a(n) + 1) \\
&= A_2(L_{m-1}) + \sum_{n<L_{m-2}} a^2(n) + 2\sum_{n<L_{m-2}} a(n) + L_{m-2} \\
&= A_2(L_{m-1}) + A_2(L_{m-2}) + 2A_1(L_{m-2}) + L_{m-2} \\
&= \frac{1}{5}[(m-2)(m-3)L_{m-3} + 5(m-2)F_{m-3} + 7(m-2)F_{m-4} + 3F_{m-2}] \\
&\quad + \frac{1}{5}[(m-3)(m-4)L_{m-4} + 5(m-3)F_{m-4} + 7(m-3)F_{m-5} + 3F_{m-3}] \\
&\quad + 2(m-2)F_{m-3} + L_{m-2} \\
&= \frac{1}{5}[(m-1)(m-2)L_{m-3} + 5(m-1)F_{m-3} + 7(m-1)F_{m-4} + 3F_{m-2}] \\
&\quad + \frac{1}{5}[(m-1)(m-2)L_{m-4} + 5(m-1)F_{m-4} + 7(m-1)F_{m-5} + 3F_{m-3}] \\
&\quad - \frac{1}{5}[2(m-1)L_{m-3} + (4m-10)L_{m-4} + 5F_{m-3} + 7F_{m-4} + 10F_{m-4} \\
&\quad + 14F_{m-5}] + 2(m-2)F_{m-3} + L_{m-2} \\
&= \frac{1}{5}[(m-1)(m-2)L_{m-2} + 5(m-1)F_{m-2} + 7(m-1)F_{m-3} + 3F_{m-1}]
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{5}\left[2(m-2)(L_{m-3}+2L_{m-4}) - 2L_{m-4} + 5(F_{m-3}+2F_{m-4})\right.\\
&\left. + 7(F_{m-4}+2F_{m-5})\right] + 2(m-2)F_{m-3} + L_{m-2}\\
&= \frac{1}{5}\left[(m-1)(m-2)L_{m-2} + 5(m-1)F_{m-2} + 7(m-1)F_{m-3} + 3F_{m-1}\right]\\
&\quad - \frac{1}{5}\left[10(m-2)F_{m-3} - 2L_{m-4} + 5L_{m-3} + 7L_{m-4}\right]\\
&\quad + 2(m-2)F_{m-3} + L_{m-2}\\
&= \frac{1}{5}\left[(m-1)(m-2)L_{m-2} + 5(m-1)F_{m-2} + 7(m-1)F_{m-3} + 3F_{m-1}\right].
\end{aligned}
$$

That is, (4) is true for $k = m$. This completes the proof of Theorem 1.

Proof of Theorem 2. Note that $N = L_{k_1} + L_{k_2} + \cdots + L_{k_s}$, applying Theorem 1 we have

$$
\begin{aligned}
A_1(N) &= \sum_{n<L_{k_1}} a(n) + \sum_{L_{k_1}\le n<N} a(n)\\
&= A_1(L_{k_1}) + \sum_{L_{k_1}\le n<N} a(n)\\
&= A_1(L_{k_1}) + \sum_{0\le n<N-L_{k_1}} a(n+L_{k_1})\\
&= A_1(L_{k_1}) + \sum_{0\le n<N-L_{k_1}} (a(n)+1)\\
&= A_1(L_{k_1}) + A_1(N-L_{k_1}) + N - L_{k_1}.
\end{aligned}
$$

and

$$
\begin{aligned}
A_2(N) &= \sum_{0\le n<L_{k_1}} a^2(n) + \sum_{L_{k_1}\le n<N} a^2(n)\\
&= A_2(L_{k_1}) + \sum_{0\le n<N-L_{k_1}} a^2(n+L_{k_1})\\
&= A_2(L_{k_1}) + \sum_{0\le n<N-L_{k_1}} (a^2(n)+2a(n)+1)\\
&= A_2(L_{k_1}) + N - L_{k_1} + A_2(N-L_{k_1}) + 2A_1(N-L_{k_1}).
\end{aligned}
$$

This proves the first part of Theorem 2.

The final formula in Theorem 2 can be proved using induction on s and the recursion formulae. This completes the proof of Theorem 2.

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On a Generalized Bisector Theorem

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In the book [1] by Smarandache (see also [2]) appears the following generalization of the well-known bisector theorem.

Let AM be a cevian of the triangle which forms the angles u and v with the sides AB and AC, respectively. Then

$$\frac{AB}{AC} = \frac{MB}{MC} \cdot \frac{\sin v}{\sin u}. \tag{76}$$

We wish to mention here that relation (1) **also** appeared in my book [3] on page 112, where it is used for a generalization of Steiner's theorem. Namely, the following result holds true (see Theorem 25 in page 112):

Let AD and AE be two cevians $(D, E \in (BC))$ forming angles α, β with the sides AB, AC, respectively. If $\widehat{A} \leq 90^\circ$ and $\alpha \leq \beta$, then

$$\frac{BD \cdot BE}{CD \cdot CE} \leq \frac{AB^2}{AC^2}. \tag{77}$$

Indeed, by applying the area resp. trigonometrical formulas of the area of a triangle, we get

$$\frac{BD}{CD} = \frac{A(ABD)}{A(ACD)} = \frac{AB \sin \alpha}{AC \sin(A-\alpha)}$$

(i.e. relation (1) with $u = \alpha$, $v = \beta - \alpha$). Similarly one has

$$\frac{BE}{CE} = \frac{AB \sin(A-\beta)}{AC \sin \beta}.$$

Therefore

$$\frac{BD \cdot BE}{CD \cdot CE} = \left(\frac{AB}{AC}\right)^2 \frac{\sin \alpha}{\sin \beta} \cdot \frac{\sin(A-\beta)}{\sin(A-\alpha)}. \tag{78}$$

Now, identity (3), by $0 < \alpha \leq \beta < 90^\circ$ and $0 < A-\beta \leq A-\alpha < 90^\circ$ gives immediately relation (2). This solution appears in [3]. For $\alpha = \beta$ one has

$$\frac{BD \cdot BE}{CD \cdot CE} = \left(\frac{AB}{AC}\right)^2 \tag{79}$$

which is the classical Steiner theorem. When $D \equiv E$, this gives the well known bisector theorem.

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On a Conjecture of F. Smarandache

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Abstract: The main purpose of this paper is to solve a problem generated by Professor F.Smarandache.

Key word: Permutation sequence; k-power.

Let n be a positive integer, n is called a k–power if $n=m^k$, where k and m are positive integer, and $k\geq 2$. Obviously, if n is a k-power, p is a prime, then we have $p^k|n$, if $p|n$.

In his book "Only Problems, not Solutions", Professor F.Smarandache defined a permutation sequence: 12, 1342, 135642, 13578642, 13579108642, 135791112108642, 1357911131412108642,13579111315161412108642,135791113151718161412108642,..., and generated a conjecture: there is no any k-power among these numbers. The main purpose of this paper is to prove that this conjecture is true.

Suppose there is a k-power a(n) among the permutation sequence. Noting the fact: $12=2^2\times 3$, we may immediately get: $a(n)\geq 1342>10000$. For the last two digits of a(n) is 42, so we have $a(n)\equiv 42(\mathrm{mod}100)$

Noting that $4|100$, we may immediately deduce : $a(n)\equiv 42\equiv 2(\mathrm{mod}4)$.

So we get $2|a(n)$, $4\nmid a(n)$. However, 2 is a prime, then $4|a(n)$ contradicts with $4\nmid a(n)$. So a(n) is not a k-power.

This complete the proof of the conjecture.

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Paradoxes Review

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Abstract: I came across one of the Smarandache divine paradoxes and felt very strongly that it is really our Buddhist's obligation to help understand the underlying truth. There seem a lot of toughest points in the cultural difference, and it will be the most difficult job to reach the mutual point as neutrality. What I can do is to try our best and find cooperation. Limited to the time, I just put a few as my first review.

Keywords: Buddhism, Daoism, Neutrosophy, Democracy, Illusion, Cause-effect, Neutrosophic Logic

1 Smarandache Divine Paradoxes at http://www.gallup.unm.edu/~smarandache/smarandache-divine-paradoxes.htm

Divine Paradox (I): Can God commit suicide?

If God cannot, then it appears that there is something God cannot do, therefore God is not omnipotent.

If God can commit suicide, then God dies - because He has to prove it, therefore God is not immortal.

1.1 There is confusion on the law of identity in logic

- There are two meanings in the referring to God: one is his eternal spirit, one is body.
- When we refer to the eternal spirit, there is no suicide at all——it is merely our illusion.
- When we refer to the body, it is actually not him, just one of his clothes. God's omnipotence implies that he can change his clothes at his will. Therefore, there is no suicide in him.

  The emperor gave a profound ceremonious funeral to Master Dharma, the First Patriarch of Buddhism in China coming from India, after his death. Weeks later however, someone just returning from the west claimed he really saw the Master going to the west. To his astonishment, the emperor decided to untomb to verify. Still to their astonishment, there is nothing more than a shining shoe in it that sent forth radiant light.

  There can be countless figures of the same Bodhisattva simultaneously exist, according to the sutras (Ven. Chin Kung [1]).

  (Since Ven. Master Chin Kung is a Buddhist monk, his Buddhist name is Chin Kung and his surname is the unified one: Sakyamuni, normally not mentioned)
- However, there is suicide as a manner we believe, so the problem becomes: whether God can show us such a manner.

1.2 God lives to save us, not to kill us

- As the greatest teacher, there is nothing silly at all in his mind. How can we compare our silliness with God?
  a) Everyone is brought up by his parents and the society with toil and moil — from pregnancy, birth, to breeding, nurturing..., they suffered everything just for the future of him. What a silly dead when he is not willing to face difficulties that his parents dealt with for decades? He must be

crazy. How can we image God as crazy? Nor even a psychologically healthy person.

b) Although God can change his clothes at his will, but he can never even think of such a manner: once we assume he showed, he would be as silly as us, and if millions of people followed this silliness, who would be the murderer?

c) The same silliness truly happens in present China where millions of innocent people believed in a cheater of Buddhism——Li Hongzhi and his Falungong.

- God signifies the supreme power and mighty of mercy.
  a) Just because compassion constitutes the nature of genius people, God never allows suicide, instead, he saves people from death.
  b) For the well-disciplined Buddhists, neither do they commit suicide, nor kill even eat animals. Instead, they often free captive animals from death.

1.3 God lives for all flesh, not for himself

- Whenever we ask whether God can show us in the manner of death? Sure, but never for himself, he suffers and dies for every being: to save us all.

1.4 The consequence of suicide is definitely shown in the life after death and in the following cycles of life as well

- Much more can we learn from Buddhism that bases its whole theory on the cause-effect phenomena of our daily routines, and the destiny lies just in this.

2. A social paradox at http://www.gallup.unm.edu/~smarandache/SocialParadox.htm

Smarandache Social Paradox:

In a democracy should the nondemocratic ideas be allowed?

a) If yes, i.e. the nondemocratic ideas are allowed, then one not has a democracy anymore. (The nondemocratic ideas may overturn the society.)

b) If no, i.e. other ideas are not allowed - even those nondemocratic -, then one not has a democracy either, because the freedom of speech is restricted.

Democracy and nondemocracy coexist in one contradiction

- There is no truth actually, just because there is prejudice.
  a) The supreme truth lies in its void nature: Dao in Daoism, the wisdom in Buddhism. Dao is void in that whenever we speak of the order of nature, what we imagine can never be Dao (inferred from Daodejing, B. Wang, X. Guo). So is Buddha: he is not shown in any kind of forms like figure, image, the truth, the ideal, etc., what we see is merely our phantasm, not real (the Diamond Sutra, Ven. Chin Kung [1]). He is ideal just because he doesn't pursuit idealness.
  b) Whenever we speak of truth, comparative to false merely, like positive to negative, good to bad, wise to error, Buddhist way to errant manner (inferred from Daodejing, B. Wang, X. Guo).
  c) There is only one step between truth and prejudice. Truth becomes prejudice when it is over believed regardless of constraint of situations.
- There is no absolute democracy.
  a) As shown above, when we mention democracy, we relate to nondemocracy too: we call for democracy because there is dictatorship.
  b) Absolute democracy has no meaning — if it had, it were self-contradictory: just as the paradox shows.

- There needs neutrality between them as balance.
  a) Absolute democracy allows self-centered societies to expand, the outcome must be conflicts. It is even worse in the expansion of heresy, e.g., in Hitler's Nazi Reich in which most of the people are cheated. The democracy turned out to be a dictatorship.
  b) There is no absolute dictatorship either. Absolute dictatorship against people's will definitely lead to its being overthrown, and that according to people's will can lead to prosperity too (in the Tang dynasty of China the emperor even invited Buddhism from India even when Chine had its own deep rooted cultures).
  c) Absolute democracy/dictatorship will definitely lead to the negation of itself. This is one of the essentialities of Daoism: things will develop in the opposite direction when they become extreme, reflected in neutrosophy as the Law of Inverse Effect (F. Smarandache).
  d) In reality society is based on the balance between these two factors, so Mao Zedong advocates the unity of democracy and centralism, i.e., democratic centralism in his theory. However he never implemented it due to some effect, e.g., he launched the Cultural Revolution.

3. **Psychology**: http://www.gallup.unm.edu/~smarandache/psychology.htm

b) Smarandache's Illusion:

Suppose you travel to a third world country, for example Romania. You arrive in the capital city of Bucharest, late in the night, and want to exchange a $100 bill to the country's currencies, which are called "lei". All exchange offices are closed. A local citizen approaches and proposes you to exchange your bill. He is a thief.

You give him the $100 bill, he gives you the equivalent in the country's currency, i.e. 25,000 lei. But the laws of the country do not allow exchange on the street, and both of you know it.

The thief cries "police!", and gives you the dollars back with one hand, while with the other hand takes back his lei, and runs out vanishing behind a building.

The thief has cheated you.

Taken by surprise, you don't realize what had happened, and looking in your hand expecting to see back a $100 bill, actually you see a $1 bill... in your mind, in the very first seconds, it appears the illusion that the $100 bill changed, under your eyes, into a $1 bill!

3.1 There is no absolute fact

One time in Tang dynasty of China, the Fifth Patriarch of Buddhism announced to his disciples that everyone write a verse to show his insight of the Buddhist wisdom.

At this, the most eligible one presented on the wall the verse:

Our body be a Bodhi tree,
Our mind a mirror bright,
Clean and polish frequently,
Let no dust alight.

Just as a choreman in the mill of the temple, Huineng answered it with his own:

There is no Bodhi tree,
Nor stand of a mirror bright,
Since all is void,
Where can the dust alight?

3.2 There is fact, but merely beliefs created by ourselves

- Let's follow the sutra (adapted from [2]):

Huineng arrived at a Temple in Guangzhou where a pennant was being blown by wind. Two monks who happened to see the pennant were debating what was in motion, the wind or the pennant.

Huineng heard their discussion and said: "It was neither the wind nor the pennant. What actually moved were your own minds." Overhearing this conversation, the assembly (a lecture was to begin) were startled at Huineng's knowledge and outstanding views.

- When we see pennant and wind we will naturally believe we are right in our consciousness, however it is subjective. In other words, what we call "the objective world" can never absolutely be objective at all.
- Whenever be believe we are objective, this belief however is subjective too.
- In fact, all these things are merely our mental creations (called illusions in Buddhism) that in turn cheat our consciousness: There is neither pennant nor wind, but our mental creations.
- The figure "you" in the paradox has created two different things: $100 currency first and then a $1 note, and he absolutely believed in both of his creations separately. As the result, he believes both are simultaneously true. But in fact **neither is true - they are all his beliefs.**
- The world is made up of our subjective beliefs that in turn cheat our consciousness. This is in fact a cumulative cause-effect phenomenon.
- Everyone can extricate himself out of this maze, said Sakyamuni and all the Buddhas, Bodhisattvas around the universe, their number is as many as that of the sands in the Ganges (Limitless Life Sutra, Ven. Chin Kung [1]).

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THE EQUATION $a^2(k+2, S(n))= a^2(k+1, S(n))+ a^2(k, S(n))$

Xigeng Chen

Abstract. For any positive integer a, let $S(a)$ be the Smarandache function of a. For any positive integers r and b, let $a(r, b)$ be the first r digits of b. In this paper we prove that the title equation has no positive integer solutions (n, k).

Key words: Smarandache function, diophantine equation

Let **N** be the set of all positive integers. For any positive integer a, let $S(a)$ be the Smarandache function of a. For any positive integer

(1) $$b=\overline{t_s\cdots t_2t_1}$$

with s digits, let

(2) $$\alpha(r,b)=\overline{t_s\cdots t_{s-r+1}}$$

be the first r digits of b. Recently, Bencze [1] proposed the following problem:

Problem Determine all solutions (n, k) of the equation

(3) $$\alpha^2(k+2,S(n))=\alpha^2(k+1,S(n))+\alpha^2(k,S(n)),n,k\in\mathbf{N}.$$

In this paper we completely solve the above-mentioned problem as follows.

Theorem The equation (3) has no solutions (n, k).

Proof. Let (n, k) be a solution of (3). It is a well known fact that $S(n)$ is a positive integer (see [2]). Let $b=S(n)$. We may assume that b

has s digits as (1). For any positive integer r, by the definition (2) of $a(r, b)$, we have

$$\alpha(r+1,b)=\begin{cases}10\alpha(r,b)+t_{s-r+1}, & \text{if } r<s,\\ \alpha(r,b), & \text{if } r\geq s.\end{cases} \tag{4}$$

If $k>s-1$, then from (4) we get $a(k+2, b)= a(k+1, b)$. Hence, by (3), we obtain $a(k, b)=0$, a contradiction.

If $k<s-1$, then from (4) we get

$$\alpha(k+2,b)\geq 10\cdot\alpha(k+1,b). \tag{5}$$

Hence, by (3) and (5), we get

$$99\cdot\alpha^2(k+1,b)\leq\alpha^2(k,b). \tag{6}$$

However, we see from (4) that $a(k+1, b)\geqslant a(k, b)$. Therefore, (6) is impossible. Thus, the equation (3) has no solutions (n, k).

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THE EQUALITY $\beta^2(k+2,S(n))=\beta^2(k+1,S(n))+\beta^2(k,S(n))$

Xingen Chen

Abstract. For any positive integer a, let $S(a)$ be the Smarandache function of a. For any positive integer r and b, let $\beta(r, b)$ be the last r digits of b. In this paper we determine all positive integer pairs (n, k) for which the title equality holds.

Key words: Smarandache function, digit, equality

For any positive integer a, let $S(a)$ be the Smarandache function of a. For any positive integer

(1) $$b=\overline{t_s\cdots t_2t_1}$$

with s digits, let

(2) $$\beta(r,b)=\overline{t_r\cdots t_1}$$

be the last r digits of b. Recently, Bencze [1] proposed the following problem:

Problem Determine all positive integer pairs (n, k) for which

(3) $$\beta^2(k+2,S(n))=\beta^2(k+1,S(n))+\beta^2(k,S(n)).$$

In this paper we completely solve the above-mentioned problem as follows.

Theorem A positive integer pair (n, k) satisfies (3) if and only if n satisfy

(4) $$S(n)=10^{k+2}c+10^{k}d,$$

where c is a nonnegative integer, d is a positive integer with $1\leqslant d\leqslant 9$.

By the definition of the Smarandache function (see [2]), we have $S(m!)=m$ for any positive integer m. Therefore, by the above theorem, we obtain the following corollary immediately.

Corollary For any fixed positive integer k, there exists infinitely many positive integers

(5) $$n=\left(10^{k+2}c+10^{k}d\right)!, c\geq 0, d=1,2,\cdots,9,$$

Satisfying (3).

The proof of Theorem Let $(n,\ k)$ be a positive integer pair satisfying (3), and let $b=S(n)$. Then b is a positive integer. We may assume that b has s digits as (1). For any positive integer r, by the definition (2) of $\beta(r,\ b)$, we have

(6) $$0\leq \beta(r,b)<10^{r}$$

and

(7) $$\beta(r+1,b)=\beta(r,b)+10^{r}t_{r+1}.$$

If $t_{k+2}\neq 0$, then from (6) and (7) we get

(8) $$\beta(k+2,b)\geq \beta(k+1,b)+10^{k+1}>\beta(k+1,b)+\beta(k,b).$$

It implies that

(9) $$\beta^{2}(k+2,b)>\beta^{2}(k+1,b)+\beta^{2}(k,b),$$

which contradicts (3).

If $t_{k+2}=0$, then from (7) we get

(10) $$\beta(k+2,b)=\beta(k+1,b).$$

Substitute (10) into (3), we get $\beta(k,\ b)=0$. It implies that $t_1=\cdots=t_k=0$

by (2). Thus, $b=S(n)$ satisfies (4). The theorem is proved.

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ON THE SMARANDACHE DOUBLE FACTORIAL FUNCTION

Maohua Le

Abstract. In this paper we discuss various problems and conjectures concerned the Smarandache double factorial function.

Keywords: Smarandache double factorial function, inequality, infinite series, infinite product, diophantine equation

For any positive integer n, the Smarandache double factorial function $Sdf(n)$ is defined as the least positive integer m such that $m!!$ is divisible by n, where

$$m!!=\begin{cases}2.4\ldots m, & \text{if } 2\mid m,\\ 1.3\ldots m, & \text{if } 2\mid m.\end{cases}$$

In this paper we shall discuss various problems and conjectures concerned $Sdf(n)$.

1. The valua of $Sdf(n)$

By the definition of $Sdf(n)$, we have $Sdf(1)=1$ and $Sdf(n)>1$ if $n>1$. We now give three general results as follows.

Theorem 1.1. If $2\nmid n$ and

$$n=p_1^{a_1}p_2^{a_2}\ldots p_k^{a_k} \tag{1.1}$$

is the factorization of n, where $p_1, p_2, \cdots, p_k$ are distinct odd primes and $a_1, a_2, \cdots, a_k$ are positive integers, then

(1.2) $$Sdf(n)=\max\left(Sdf\left(p_1^{a_1}\right),Sdf\left(p_2^{a_2}\right),\ldots,Sdf\left(p_k^{a_k}\right)\right).$$

Proof. Let $m_i=sdf\left(P_i^{a_i}\right)$ for i=1, 2, ⋯, k. Then we get $2\nmid m_i$ (i=1, 2, ⋯, k) and

(1.3) $$p_i^{a_i}\mid m_i!!, i=1,2,\ldots,k.$$

Furthur let m=max(m_1, m_2, ⋯, m_k). Then we have

(1.4) $$m_i!!\mid m!!,\ i=1,2,\ \cdots,\ k.$$

Therefore, by (1.3) and (1.4), we get

(1.5) $$p_i^{a_i}\mid m!!, i=1,2,\ldots,k.$$

Notice that p_1, p_2, ⋯, p_k are distinct odd primes. We have

(1.6) $$\gcd(p_i^{a_i},p_j^{a_j})=1, 1\le i<j\le k.$$

Thus, by (1.1), (1.5) and (1.6), we obtain $n\mid m!!$. It implies that

(1.7) $$Sdf(n)\leqslant m.$$

On the other hand, by the definition of m, if $Sdf(n)<m$, then there exists a prime power $p_j^{a_j}\left(1\le j\le k\right)$ such that

(1.8) $$p_j^{a_j}\mid Sdf(n)!!.$$

By (1.1) and (1.8), we get $n\mid Sdf(n)!!$, a contradiction. Therefore, by (1.7), we obtain $Sdf(n)=m$. It implies that (1.2) holds. The theorem is proved.

Theorem 1.2. If $2\mid n$ and

(1.9) $$n=2^a n_1,$$

where a, n_1 are positive integers with $2\nmid n_1$, then

(1.10) $$Sdf(n)\leqslant\max(Sdf(2^a), 2Sdf(n_1)).$$

Proof. Let $m_0=Sdf(2^a)$ and $m_1=Sdf(n_1)$. Then we have

(1.11) $$2^a|m_0!!,\ n_1|m_1!!.$$

Since $(2m_1)!!=2.4\cdots(2m_1)=2^{m_1}\cdot m_1!=2^{m_1}\cdot m_1!!(m_1-1)!!$, we get $m_1!!|(2m_1)!!$. It implies that

(1.12) $$n_1|(2m_1)!!.$$

Let $m=\max(m_0, 2m_1)$. Then we have $m_0!!|m!!$ and $(2m_1)!!|m!!$. Since gcd $(2^a, n_1)=1$, we see from (1.9), (1.11) and (1.12) that $n|m!!$. Thus, we obtain $Sdf(n)\leqslant m$. It implies that (1.10) holds. The theorem is proved.

Theorem 1.3. Let a, b be two positive integers. Then we have

(1.13) $$Sdf(ab)\leq\begin{cases} Sdf(a)+Sdf(b), & \text{if } 2|a \text{ and } 2|b,\\ Sdf(a)+2Sdf(b), & \text{if } 2|a \text{ and } 2|b,\\ 2Sdf(a)+2Sdf(b)-1, & \text{if } 2|a \text{ and } 2|b. \end{cases}$$

Proof. By Theorem 4.13 of [4], if $2|a$ and $2|b$, then

(1.14) $$Sdf(a)=2r,\ Sdf(b)=2s,$$

where r, s are positive integers. We see from (1.14) that

(1.15) $$a|(2r)!!,\ b|(2s)!!.$$

Notice that

(1.16) $$\frac{(2r+2s)!!}{(2r)!!(2s)!!}=\frac{2^{r+s}\cdot(r+s)!}{(2^r\cdot r!)(2^s\cdot s!)}=\frac{(r+s)!}{r!s!}=\binom{r+s}{r},$$

where $\binom{r+s}{r}$ is a binomial coefficient. Since $\binom{r+s}{r}$ is a positive integer, we see from (1.16) that

(1.17) $(2r)!!(2s)!!|(2r+2s)!!$

Thus, by (1.15) and (1.17), we get $ab|(2r+2s)!!$. It implies that

(1.18) $Sdf(ab)\leqslant 2r+2s$, if $2|a$ and $2|b$.

If $2|a$ and $2\nmid b$, then

(1.19) $Sdf(a)=2r, Sdf(b)=2s+1,$

where a is a positive integer and s is a nonnegative integer. By (1.19), we get

(1.20) $a|(2r)!!, b|(2s+1)!!.$

Notice that

$$(1.21)\qquad \frac{(2r+4s+2)!!}{(2r)!!(2s+1)!!}=\frac{2^{r+2s+1}\cdot (r+2s+1)!}{2^{r}\cdot r!}\cdot\frac{2^{s}\cdot s!}{(2S+1)!}$$
$$=2^{3s+1}\cdot s!\frac{(r+2s+1)!}{r!(2s+1)}=2^{3s-1}\cdot s!\binom{r+2s+1}{r}.$$

We find from (1.21) that

(1.22) $(2r)!!(2s+1)!!|(2r+4s+2)!!.$

Thus, by (1.20) and (1.22), we obtain $ab|(2r+4s+2)!!$. It implies that

(1.23) $Sdf(ab)\leqslant 2r+4s+2$, if $2|a$ and $2|b$.

If $2 \mid a$ and $2 \mid b$, then

(1.24) $Sdf(a)=2r+1, Sdf(b)=2s+1,$

where r, s are nonnegative integers. By (1.24), we get

(1.25) $a|(2r+1)!!, b|(2s+1)!!.$

Notice that

$$(1.26)\quad \frac{(4r+4s+3)!!}{(2r+1)!!(2s+1)!!}=\frac{(4r+4s+3)!}{(4r+4s+2)!!}\cdot\frac{(2r)!!}{(2r+1)!}\cdot\frac{(2s)!!}{(2s+1)!}$$
$$=\frac{(4r+4s+3)!}{2^{2r+2s+1}\cdot(2r+2s+1)!}\cdot\frac{2^r\cdot r!}{(2r+1)!}\cdot\frac{2^s\cdot s!}{(2s+1)!}$$
$$=\frac{r!s!}{2^{r+s+1}}\binom{4r+4s+3}{2r+2s+1,2r+1,2s+1},$$

where $\binom{4r+4s+3}{2r+2s+1,2r+1,2s+1}$ is a polynomial coefficient. Since $\binom{4r+4s+3}{2r+2s+1,2r+1,2s+1}$ is a positive integer and $(2r+1)!!$, $(2s+1)!!$ are odd integers, we see from (1.26) that

$$(1.27)\qquad (2r+1)!!(2s+1)!!\mid(4r+4s+3)!!.$$

Thus, by (1.25) and (1.27), we get $ab\mid(4r+4s+3)!!$. It implies that

$$(1.28)\qquad Sdf(ab)\leqslant 4r+4s+3, \text{ if } 2\mid a \text{ and } 2\mid b.$$

The combination of (1.18), (1.23) and (1.28) yields (1.13). The n theorem is proved.

Theorem 1.4 Let p be a prime and let a be a positive integer. The we have

$$(1.29)\qquad p\mid Sdf(p^a).$$

Proof. Let $m=Sdf(p^a)$. By Theorem 4.13 of [4], if $p=2$, then m is even. Hence, (1.29) holds for $p=2$. If $p>2$, then m is an odd integer with

$$(1.30)\qquad p^a\mid m!!$$

We now suppose that $p \mid m$. Let t be the greatest odd integer such that $t < m$ and $p|t$. Then we have

(1.31) $$m!!=t!!(t+2)\cdots(m-2)m,$$

where $t+2, \cdots, m-2, m$ are integers satisfying $p \nmid (t+2)\cdots(m-2)m$. Therefore, by (1.30) and (1.31), we get

(1.32) $$p^a | t!!$$

By (1.32), we get $m=Sdf(p^q) \leqslant t < m$, a contradiction. Thus, we obtain $p|m$. The theorem is proved.

Theorem 1.5 Let p be the least prime divisor of n. Then we have

(1.33) $$Sdf(n) \geqslant p.$$

Proof. Let $m=Sdf(n)$. By Theorem 4.13 of [4], if $2|n$, then $p=2$ and m is an even integer. So we get (1.33).

If $2 \mid n$, let $n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$, where $p_1, p_2, \cdots, p_k$ are distinct odd primes and $a_1, a_2, \cdots, a_k$ are positive integers. By Theorem 1.1, we get

(1.34) $$m = \max\left(Sdf\left(p_1^{a_1}\right), Sdf\left(p_2^{a_2}\right), \ldots, Sdf\left(p_k^{a_k}\right)\right)$$

Further, by Theorem 1.4, we have $p_i \mid Sdf\left(p_i^{a_i}\right)$ for $i=1, 2, \cdots, k$. It implies that $Sdf\left(p_i^{a_i}\right) \geq p_i$ for $i=1, 2, \cdots, k$. Thus, by (1.34), we obtain

(1.35) $$m \geqslant \min(p_1, p_2, \cdots, p_k)=p.$$

The theorem is proved.

Theorem 1.6 For any positive integer n, we have

(1.36) $$Sdf(n!)=\begin{cases} n, & \text{if } n=1,2, \\ 2n, & \text{if } n>2. \end{cases}$$

Proof. Let $m=Sdf(n!)$. Then (1.36) holds for $n=1, 2$. If $n>2$, then both $n!$ and m are even. Since $(2n)!!=2^n n!$, we get

(1.37) $$m \leqslant 2n.$$

If $m<2n$, then $m=2n-2r$, where r is a positive integer. Since $m=Sdf(n!)$,

(1.38) $$\frac{(2n-2r)!!}{n!}=\frac{2^{n-r}\cdot (n-r)!}{n!}=\frac{2^{n-r}}{(n-r+1)\ldots(n-1)n}$$

must be an integer. But, since either n-1 or n is an odd integer great than 1, it is impossible by (1.38). Thus, by (1.37), we obtain $m=2n$. The theorem is proved.

Theorem 1.7 The equality

(1.39) $$Sdf(n)=n$$

holds if and only if n satisfies one of the following conditions:

(i) $n=1, 9$.

(ii) $n=p$, where p is a prime.

(iii) $n=2p$, where p is a prime.

Proof. Let $m=Sdf(n)$. If $2 \mid n$, let $n=p_1^{a_1}p_2^{a_2}\ldots p_k^{a_k}$ be the factorization of n. By Theorem 1.1, we (1.34). Further, by Theorem 4.7 of [4], we have

(1.40) $$Sdf\left(p_i^{a_i}\right)\leq p_i^{a_i}, i=1,2,\ldots,k.$$

Therefore, by (1.34) and (1.40), we obtain

(1.41) $$m \leq \max\left(p_1^{a_1}, p_2^{a_2},\ldots, p_k^{a_k}\right)$$

It implies that if $k>1$, then $m<n$. If $k=1$ and (1.39) holds, then

(1.42) $$m = Sdf\left(p_1^{a_1}\right) = p_1^{a_1}.$$

By Theorem 4.1 of [4], (1.42) holds for a_1=1. Since $2 \mid n$, p_1 is an odd prime. By Theorem 1.3, if (1.42) holds, then we have

(1.43) $$p_1^{a_1} = m = Sdf\left(p_1^{a_1}\right) = Sdf(p_1 p_1 \ldots p_1) \le 2a_1 \cdot Sdf(p_1) - 1 = 2a_1 p_1 - 1$$

Since $p_1 \geqslant 3$, (1.43) is impossible for $a_1 > 2$. If a_1=2, then from (1.43) we get

(1.44) $$p_1^2 \leqslant 4p_1 - 1,$$

whence we obtain p_1=3. Thus, (1.39) holds for an odd integer n if and only if n=1.9 or p, where p is an odd prime.

If $2|n$, then n can be rewritten as (1.9), where n_1 is an odd integer with $n_1 \geqslant 1$. By Theorem 1.2, if (1.39) holds, then we have

(1.45) $$n = 2^a n_1 \leqslant \max(Sdf(2^a), 2Sdf(n_1)).$$

We see from (1.45) that if (1.39) holds, then either n_1=1 or a=1.

When n_1=1, we get from (1.39) that a=1 or 2. When a=1, we get,

(1.46) $$2n_1 = Sdf(2n_1).$$

It is a well known fact that if n_1 is not an odd prime, then there exists a positive integer t such that $t < n_1$ and $n_1 \mid t!$. Since $(2t)!! = 2^t \cdot t!$, we get

(1.47) $$Sdf(2n_1) \leqslant 2t \leqslant 2n_1,$$

a contradiction. Therefore, n_1 must be an odd prime. In this case, if $Sdf(2n_1) < 2n_1$, then $Sdf(2n_1) = 2n_1 - 2r$, where r is a positive integer. But, since

(1.48) $$\frac{(2n_1 - 2r)!!}{2n_1} = \frac{2^{n_1 - r} \cdot (n_1 - r)}{2n_1} = \frac{2^{n_1 - r - 1} \cdot (n_1 - r)!}{n_1}$$

is not an integer, it is impossible. Thus, (1.39) holds for an even

integer if and only if $n=2p$, where p is a prime. The theorem is proved.

2. The inequalities concerned $Sdf(n)$

Let n be a positive integer. In [4], Russo posed the following problems and conjectures.

(2.1) $$\frac{n}{Sdf(n)} \leq \frac{n}{8} + 2$$

Problem 2.1. Is the inequality

true for any n?

Problem 2.2. Is the inequality

(2.2) $$\frac{Sdf(n)}{n} > \frac{1}{n^{0.73}}$$

true for any n?

Problem 2.3. Is the inequality

(2.3) $$\frac{1}{n \cdot Sdf(n)} < n^{-5/4}$$

true for any n?

(2.4) $$\frac{1}{n} + \frac{1}{Sdf(n)} < n^{-1/4}$$

Problem 2.4. Is the inequality

true for any n with $n>2$?

Conjecture 2.1. For any positive number ε, there exist some n such that

(2.5) $$\frac{Sdf(n)}{n} < \varepsilon$$

In this respect, Russo [4] showed that if $n \leqslant 1000$, then the

inequalities (2.1), (2.2), (2.3) and (2.4) are true. We now completely solve the above-mentioned questions as follows.

Theorem 2.1. For any positive integer n, the inequality (2.1) is true.

Proof. We may assume that $n>1000$. Since $m!!\leqslant 945$ for $m=1$, 2, $\cdots$, 9, if $n>1000$, then $Sdf(n)\geqslant 10$. So we have

$$\frac{n}{Sdf(n)}\leq\frac{n}{10}<\frac{n}{8}+2. \tag{2.6}$$

It implies that (2.1) holds. The theorem is proved.

The above theorem shows that the answer of Problem 2.1 is "yes".

In order to solve Problems 2.2, 2.3 and 2.4, we introduce the following result.

Theorem 2.2. If $n=(2r)!!$, where r is a positive integer withr $r\geqslant 20$, then

$$Sdf(n)<n^{0.1}. \tag{2.7}$$

Proof. We now suppose that

$$Sdf(n)\geqslant n^{0.1}. \tag{2.8}$$

Since $n=(2r)!!$, we get $Sdf(n)=2r$. Substitute it into (2.8), we obtain that if $r\geqslant 20$, then

$$2r\geqslant((2r)!!)^{0.1}=2^{0.1r}(r!)^{0.1}\geqslant 2^2(r!)^{0.1}. \tag{2.9}$$

By the Strling theorem (see [1]), we have

$$r!>\sqrt{2\pi r}\left(\frac{r}{e}\right)^r. \tag{2.10}$$

Since $r\geqslant 20$, we get $r/e>\sqrt{r}$. Hence, by (2.9) and (2.10), we obtain

(2.11) $$2r \geqslant 4(r!)^{0.1} > 4r^{0.05r} \geqslant 4r,$$

a contradiction. Thus, we get (2.7). The theorem is proved.

By the above theorem, we obtain the following corollary immediately.

Corollary 2.1. If $n=(2r)!!$, where r is a positive integer with $r \geqslant 20$, then the inequalities (2.2), (2.3) and (2.4) are false.

The above corollary means that the answers of Problems 2.2, 2.3 and 2.4 are "no".

Theorem 2.3. For any positive number ε, there exist some positive integers n satisfy (2.5).

Proof. Let $n=(2r)!!$, where r is a positive integer with $r \geqslant 20$. By Theorem 2.2, we have

(2.12) $$\frac{Sdf(n)}{n} < \frac{n^{0.1}}{n} = \frac{1}{n^{0.9}}.$$

By (2.12), we get

(2.13) $$\lim_{r \to \infty} \frac{Sdf(n)}{n} = 0.$$

Thus, by (2.13), the theorem is proved.

By the above theorem, we see that Conjecture 2.1 is true.

3. The difference $|Sdf(n+1)-Sdf(n)|$

In [4], Russo posed the following problem.

Problem 3.1. Is the difference $|Sdf(n+1)-Sdf(n)|$ bounded or unbounded?

We now solve this problem as follows.

Theorem 3.1. The difference $|Sdf(n+1)-Sdf(n)|$ is unbounded.

Proof. Let m be a positive integer, and let p be a prime. Further let ord $(p, m!)$ denote the order of p in m. For any positive integer a, it is a well known fact that

$$\text{ord}(p,a!)=\sum_{k=1}^{\infty}\left[\frac{a}{pk}\right]. \tag{3.1}$$

(see Theorem 1.11.1 of [3]).

Let r be a positive integer. Then we have

$$2^r!!=2\cdot 4\cdots 2^r=2^s\cdot 2^{r-1}!, \tag{3.2}$$

where

$$s=2^{r-1}. \tag{3.3}$$

By (3.1), (3.2) and (3.3), we get

$$\text{ord}(2, 2^r!!)=2^{r-1}+\text{ord}(2, 2^{r-1}!)=2^{r-1}+(2^{r-2}+\cdots+2+1)=2^r-1 \tag{3.4}$$

Let $n=2^t$, where $t=2^r$. Then, by (3.4), we get

$$Sdf(n)=2^r+2 \tag{3.5}$$

On the other hand, then $n+1=2^t+1$ is a Fermat number. By the proof of Theorem 5.12.1 of [3], every prime divisor q of $n+1$ is the form $q=2^{r+1}l+1$, where l is a positive integer. It implies that

$$q\geqslant 2^{r+1}+1. \tag{3.6}$$

Since $n+1$ is an odd integer, by Theorem 1.4, we get from (3.6) that

$$Sdf(n+1)\geqslant q\geqslant 2^{r+1}+1. \tag{3.7}$$

We see from (3.8) that the difference $|Sdf(n+1)-Sdf(n)|$ is unbounded. Thus, the theorem is proved.

4. Some infinite series and products concerned $Sdf(n)$

In [4], Russo posed the following problems.

Problem 4.1. Evaluate the infinite series

$$S=\sum_{n=1}^{\infty}\frac{(-1)^n}{Sdf(n)}. \tag{4.1}$$

Problem 4.2. Evaluate the infinite product

$$P=\prod_{n=1}^{\infty}\frac{1}{Sdf(n)}. \tag{4.2}$$

We now solve the above-mentioned problems as follows.

Theorem 4.1. $S=\infty$.

Proof. For any nonnegative integer m, let

$$g(m)=\frac{-1}{Sdf(2m+1)}+\sum_{i=1}^{\infty}\frac{1}{Sdf\left(2^i(2m+1)\right)}. \tag{4.3}$$

By (4.1) and (4.3), we get

$$S=\sum_{m=0}^{\infty}g(m). \tag{4.4}$$

We see from (4.3) that

$$\begin{aligned} g(0)&=-1+\frac{1}{Sdf(2)}+\frac{1}{Sdf(4)}+\frac{1}{Sdf(8)}+\cdots \\ &=-1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{6}+\cdots>\frac{1}{6}. \end{aligned} \tag{4.5}$$

For positive integer m, let $t=Sdf(2m+1)$. Then t is an odd integer with $t\geqslant 3$. Notice that $2m+1|t!!$ and

$$(2t)!!=2^t\cdot t!!. \tag{4.6}$$

We get from (4.6) that $2^j(2m+1)|(2t)!!$ for $j=1, 2, \cdots, t$. It implies that

(4.7) $$Sdf(2^j(2m+1))\leqslant 2t, j=1, 2, \cdots, t.$$

Therefore, by (4.3) and (4.7), we obtain

(4.8) $$g(m)>-\frac{1}{t}+\frac{1}{2t}+\frac{1}{2t}+\frac{1}{2t}=\frac{1}{2t}.$$

On the other hand, by Theorem 4.7 of [4], we have $t\leqslant 2m+1$. By (4.8), we get

(4.9) $$g(m)>\frac{1}{2(2m+1)}.$$

Thus, by (4.4), (4.5) and (4.9), we obtain

(4.10) $$S>\frac{1}{6}+\sum_{m=1}^{\infty}\frac{1}{2(2m+1)}=\infty.$$

The theorem is proved.

Theorem 4.2. $P=0$.

Proof. Since $Sdf(n)>1$ if $n>1$, by (4.2), we get $p=0$ immediately. The theorem is proved.

5. The diophantine equations concerned $Sdf(n)$

Let **N** be the set of all positive integers. In [4], Russo posed the following problems.

Problem 5.1 Find all the solutions n of the equation

(5.1) $$Sdf(n)!=Sdf(n!), n\in\mathbf{N}.$$

Problem 5.2 Is the equation

(5.2) $$(Sdf(n))^k=k\cdot Sdf(nk), n, k\in\mathbf{N}, n>1, k>1$$

have solutions (n, k)?

Problem 5.3 Is the equation

(5.3) $$Sdf(mn)=m^k \cdot Sdf(m),\ m, n, k\in \mathbf{N}$$

have solutions (m, n, k)?

We now completely solve the above-mentioned problems as follows.

Theorem 5.1 The equation (5.1) has only the solutions n=1, 2, 3.

Proof. Clearly, (5.1) has solutions n=1, 2, 3. We suppose that (5.1) has a solution n with $n>3$. By Theorem 1.6, if $n>2$, then

(5.4) $$Sdf(n)!=2n.$$

Substitute (5.4) into (5.1), we get

(5.5) $$Sdf(n)!=2n.$$

Let $m=Sdf(n)$. If $n>3$ and $2 \mid n$, then $n\geqslant 5$, $m\geqslant 5$ and $4|m!$. However, since $2\|2n$, (5.5) is impossible.

If $n>3$ and $2|n$, then $m=2t$, where t is a positive integer with $t>1$. From (5.5), we get

(5.6) $$(2t)!=2n.$$

Since $m=Sdf(n)$, we have $n|(2t)!!$. It implies that

$$\frac{(2t)!!}{n}=\frac{2\cdot(2t)!!}{(2t)!}=\frac{2\cdot(2t)!!}{(2t)!!(2t-1)!!}=\frac{2}{(2t-1)!!}$$

must be an integer. But, since $t>1$, it is impossible. Thus, (5.1) has no solutions n with $n>3$. The theorem is proved.

Theorem 5.2 The equation (5.2) has only the solutions (n, k)=(2, 4) and (3, 3).

Proof. Let (n, k) be a solution of (5.2). Further, let $m=Sdf(n)$. By Theorem 1.3, we get

(5.7) $$Sdf(nk)<2 \cdot Sdf(n)+2 \cdot Sdf(k)\geqslant 2(m+k).$$

Hence, by (5.2) and (5.7), we obtain

(5.8) $$m^k<2k(m+k),\ m>1,\ k>1.$$

If $m=2$, then from (5.8) we get $k\leqslant 6$. Notice that $n=2$ if $m=2$. We find from (5.2) that if $m=2$ and $k\leqslant 6$, then (5.2) has only the solution $(n, k)=(2, 4)$

If $m=3$, then from (5.8) we get $k\leqslant 3$. Since $n=3$ if $m=3$. We see from (5.2) that if $m=2$ and $k\leqslant 3$, then (5.2) has only the solution $(n, k)=(3, 3)$

If $m=4$, then from (5.8) we get $k\leqslant 2$. Notice that $n=4$ or 8 if $m=4$ and $n=5$ or 15 if $m=5$. Then (5.2) has no solution (n, k). Thus, (5.2) has only the solutions $(n, k)=(2, 4)$ and (3.3). The theorem is proved.

Theorem 5.3. All the solutions (m, n, k) of (5.3) are given in the following four classes:

(i) $m=1$, n and k are positive integers.

(ii) $n=1$, $k=1$, $m=1, 9, p$ or 2p, where p is a prime.

(iii) $m=2$, $k=1$, n is 2 or an odd integer with $n\geqslant 1$.

(iv) $m=3$, $k=1$, $n=3$.

Proof. If $m=1$, then (5.3) holds for any positive integers n and k. By Theorem 1.7, if $n=1$, then from (5.3) we get (ii). Thus, (i) and (ii) are proved.

Let (m, n, k) be a solution of (5.3) satisfying $m>1$ and $n>1$. By Theorem 1.3, if $2|m$ and $2|n$, then we have

(5.9) $$Sdf(mn)\leqslant Sdf(m)+Sdf(n).$$

Further, by Theorem 4.7 of [4], $Sdf(m)\leqslant m$. Therefore, by (5.3) and (5.9), we obtain

(5.10) $$m\geqslant (m^k-1)Sdf(n).$$

When $n=2$, we get from (5.10) that $m=2$ and $k=1$.

When $n>2$, we get $Sdf(n)\geqslant 4$ and (5.10) is impossible.

If $2|m$ and $2 \nmid n$, then

(5.11) $$Sdf(mn)\leqslant Sdf(m)+2\cdot Sdf(n).$$

Notice that $m\geqslant 2$, n is an odd integer with $n\geqslant 3$, $Sdf(n)\geqslant 3$. We obtain from (5.3) and (5.11) that

(5.12) $$m\geqslant Sdf(m)\geqslant (m^k-2)Sdf(n)\geqslant 3(m^k-2)\geqslant 3(m-2).$$

From (5.12), we get $m=2$. Then, by (5.3), we obtain

(5.13) $$Sdf(2n)=2^k\cdot Sdf(n).$$

Since $Sdf(2n)\leqslant 2n$, we see from (5.13) that $k=1$ and

(5.14) $$Sdf(2n)=2\cdot Sdf(n).$$

Notice that (5.14) holds for any odd integer n with $n\geqslant 1$. We get (iii).

If $2 \nmid m$ and $2|n$, then we have

(5.15) $$Sdf(mn)\leqslant 2\cdot Sdf(m)+Sdf(n).$$

By (5.3) and (5.15), we get

(5.16) $$2m\geqslant 2\cdot Sdf(m)\geqslant (m^k-1)\cdot Sdf(n).$$

When $n=2$, we see from (5.16) that $m=3$ and $k=1$. When $n>2$, we get from (5.16) that $2m\geqslant 4(m^k-1)\geqslant 4(m-1)>2m$, a contradiction.

If $2 \mid m$ and $2 \mid n$, then we have

(5.17) $$Sdf(mn) \leqslant 2 \cdot Sdf(m)+2 \cdot Sdf(n)-1.$$

By (5.3) and (5.17), we get

(5.18) $$2m-1 \geqslant 2 \cdot Sdf(m)-1 \geqslant (m^k-2) \cdot Sdf(n) \geqslant 3(m^k-2).$$

It implies that $k=1$ and $m=3$ or 5. Wher $m=3$ and $k=1$, we get from (5.3) that

(5.19) $$Sdf(3n)=3 \cdot Sdf(n).$$

Since $Sdf(3n) \leqslant Sdf(n)+6$, we find from (5.19) that $n=3$. Thus, we get (iv). When $m=5$ and $k=1$, we have

(5.20) $$Sdf(5n)=5 \cdot Sdf(n).$$

Since $Sdf(5n) \leqslant Sdf(n)+10$, (5.20) is impossible. To sum up, the theorem is proved.

Let p be a prime, and let $N(p)$ denote the number of solutions x of the equation

(5.21) $$Sdf(x)=p, x \in \mathbf{N}.$$

Recently, Johnson showed that if p is an odd prime, then

(5.22) $$N(p)=2^{(p-3)/2}.$$

Unfortunately, the above-mentioned result is false. For example, by (5.22), we get $N(19)=2^8=256$. However, the fact is that $N(19)=240$. We now give a general result as follows.

Theorem 5.4. For any positive integer t, let $p(t)$ denote the tth odd prime. If $p=p(t)$, then

(5.23) $$N(p)=\prod_{i=1}^{t-1}(a(i)+1),$$

where

(5.24) $$a(i)=\sum_{m=1}^{\infty}\left(\left[\frac{p-2}{(p(i))^m}\right]-\left[\frac{(p-3)/2}{(p(i))^m}\right]\right), i=1,2,\ldots,t-1.$$

Proof. Let x be a solution of (5.21). It is an obvious fact that

(5.25) $$x=dp.$$

where d is a divisor of $(p\text{-}2)!!$. So we have

(5.26) $$N(p)=d((p\text{-}2)!!),$$

where $d((p\text{-}2)!!)$ is the number of distinct divisors d of $(p\text{-}2)!!$.

By the definition of $(p\text{-}2)!!$, we have

(5.27) $$(p\text{-}2)!!=(p(1))^{a(1)}(p(2))^{a(2)}\cdots(p(t\text{-}1))^{a(t-1)},$$

where

(5.28) $$a(i)=\mathrm{ord}(p(i),(p\text{-}2)!!),\ i=1,2,\cdots,t\text{-}1.$$

Notice thet

(5.29) $$(p-2)!!=\frac{(p-2)!}{2^{(p-3)/2}\cdot\left(\frac{p-3}{2}\right)!}.$$

We get

(5.30) $$\mathrm{ord}(p(i),(p-2)!!)=\mathrm{ord}(p(i),(p-2)!)-\mathrm{ord}\left(p(i),\left(\frac{p-3}{2}\right)!\right),$$

Therefore, by Theorem 1.11.1 of [3], we see from (5.28) and (5.30) that $a(i)$ (i=1, 2, $\cdots$, t-1) satisfy (5.24). Further, by Theorem 273 of [2], we get from (5.27) that

(5.31) $$d((p-2)!!) = \prod_{i=1}^{t-1}(a(i)+1).$$

Thus, by (5.26), we obtain (5.23). The theorem is proved.

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ON THE PSEUDO-SMARANDACHE SQUAREFREE FUNCTION

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Abstract. In this paper we discuss various problems and conjectures concered the pseudo-Smarandache squarefree function.

Keywords: pseudo-Smarandache squarefree function, difference, infinite series, infinite product, diophantine equation

For any positive integer n, the pseudo-Smarandache squarefree function $ZW(n)$ is defined as the least positive integer m such that m^n is divisible by n. In this paper we shall discuss various problems and conjectures concered $ZW(n)$.

1. The value of $ZW(n)$

By the definition of $ZW(n)$, we have $ZW(1)=1$. For $n>1$, we give a general result as follows.

Theoren 1.1. If $n>1$, then $ZW(n)=p_1p_2\cdots p_k$, where $p_1, p_2, \cdots, p_k$ are distinct prime divisors of n.

Proof. Let $m=ZW(n)$. Let $p_1, p_2, \cdots, p_k$ be distinct prime divisors of n. Since $n|m^n$, we get $p_i|m$ for $i=1, 2, \cdots, k$. It implies that $p_1p_2\cdots p_k|m$ and

$$m\geqslant p_1p_2\cdots p_k. \tag{1.1}$$

On the other hand, let $r(i)$ ($i=1, 2, \cdots, k$) denote the order of p_i ($i=1, 2, \cdots, k$) in n. Then we have

(1.2) $$r(i) \le \frac{\log n}{\log p_i} < n, i = 1,2,\ldots,k.$$

Thus, we see from (1.2) that $(p_1p_2\cdots p_k)^n$ is divisible by n. It implies that

(1.3) $$m \leqslant p_1p_2\cdots p_k.$$

The combination of (1.1) and (1.3) yields $m = p_1p_2\cdots p_k$. The theorem is proved.

2. The difference $|ZW(n+1)-ZW(n)|$

In [3], Russo given the following two conjectures.

Conjecture 2.1. The difference $|ZW(n+1)-ZW(n)|$ is unbounded.

Conjecture 2.2. $ZW(n)$ is not a Lipschitz function.

In this respect, Russo [3] showed that if the Lehmer-Schinzel conjecture concered Fermat numbers is true (see [2]), then Conjectures 2.1 and 2.2 are true. However, the Lehmer-Schinzel conjecture is not resolved as yet. We now completely verify the above-mentioned conjectures as follows.

Theorem 2.1. The difference $|ZW(n+1)-ZW(n)|$ is unbounded.

Proof. Let p be an odd prime. Let $n=2^p-1$, and let q be a prime divisor of n. By a well known result of Birkhoff and Vandiver [1], we have $q=2lp+1$, where l is a positive integer. Therefore, by Theorem 1.1, we get

(2.1) $$ZW(n)=ZW(2^p-1) \geqslant q=2lp+1 \geqslant 2p+1.$$

On the other hand, apply Theorem 1.1 again, we get

(2.2) $$ZW(n+1)=ZW(2^p)=2.$$

By (2.1) and (2.2), we obtain

(2.3) $$|ZW(n+1)-ZW(n)| \geqslant 2p-1.$$

Notice that there exist infinitely many odd primes p. Thus, we find from (2.3) that the difference $|ZW(n+1)-ZW(n)|$ is unbounded. The theorem is proved.

As a direct consequence of Theorem 2.1, we obtain the following corollary.

Corollary 2.1. $ZW(n)$ is not a Lipschitz function.

3. The sum and product of the reciprocal of $ZW(n)$

Let $\mathbf{R}$ be the set of all real numbers. In [3], Russo posed the following two problems.

Problem 3.1. Evaluate the infinite product

(3.1) $$P = \prod_{n=1}^{\infty} \frac{1}{ZW(n)}.$$

Problem 3.2. Study the convergence of the infinite series

(3.2) $$S(a) = \sum_{n=1}^{\infty} \frac{1}{(ZW(n))^a}, a \in \mathbf{R}, a > 0.$$

We now completely solve the above-mentioned problems as follows.

Theorem 3.1. $P=0$.

Proof. By Theorem 1.1, we get $ZW(n)>1$ for any positive integer n with $n>1$. Thus, by (3.1), we obtain $P=0$ immediately. The theorem is proved.

Theorem 3.2. For any positive number a, $S(a)$ is divergence.

Proof. we get from (3.1) that

$$S(a)=\sum_{n=1}^{\infty}\frac{1}{(ZW(n))^a}>\sum_{r=1}^{\infty}\frac{1}{(ZW(2^r))^a}. \tag{3.3}$$

By Theorem 1.1, we have

$$ZW(2^r)=2\ , \tag{3.4}$$

for any positive integer r. Substitute (3.4) into (3.3), we get

$$S(a)=\sum_{r=1}^{\infty}\frac{1}{2^r}=\infty. \tag{3.5}$$

We find from (3.5) that $S(a)$ is divergence. The theorem is proved.

4. Diophantine equations concerning $ZW(n)$

Let **N** be the set of all positive integers. In [3], Russo posed the following problems concerned diophantine equations.

Problem 4.1. Find all solutions n of the equation

$$ZW(n)=ZW(n+1)ZW(n+2),\ n\in\mathbf{N}. \tag{4.1}$$

Problem 4.2. Solve the equation

$$ZW(n).\ ZW(n+1)=ZW(n+2),\ n\in\mathbf{N}. \tag{4.2}$$

Problem 4.3. Solve the equation

$$ZW(n).\ ZW(n+1)=ZW(n+2).\ ZW(n+3),\ n\in\mathbf{N}. \tag{4.3}$$

Problem 4.4. Solve the equation

$$ZW(mn)=m^kZW(n),\ m,\ n,\ k\in\mathbf{N}. \tag{4.4}$$

Problem 4.5. Solve the equation

$$(ZW(n))^k=k.\ ZW(kn),\ k,\ n\in\mathbf{N},\ k>1,\ n>1. \tag{4.5}$$

Problem 4.6. Solve the equation

$$(ZW(n))^k+(ZW(n))^{k-1}+\cdots+ZW(n)=n,\ k,\ n\in\mathbf{N},\ k>1. \tag{4.6}$$

In this respect, Russo [3] showed that (4.1), (4.2) and (4.3) have

no solutions n with $n \leqslant 1000$, and (4.6) has no solutions (n, k) satisfying $n \leqslant 1000$ and $k \leqslant 5$. We now completely solve the above-mentioned problems as follows.

Theorem 4.1. The equation (4.1) has no solutions n.

Proof. Let n be a solution of (4.1). Further let p be a prime divisor of $n+1$. By Theorem 1.1, we get $p|ZW(n+1)$. Therefore, by (4.1), we get $p|ZW(n)$. It implies that p is also a prime divisor of n. However, since gcd $(n, n+1)=1$, it is impossible. The theorem is proved.

By the same method as in the proof of Theorem 4.1, we can prove the following theorem without any difficult.

Theorem 4.2. The equation (4.2) has no solutions n.

Theorem 4.3. The equation (4.3) has no solutions n.

Proof. Let n be a solution of (4.3). Further let $p_1, p_2, \cdots, p_k$ and $q_1, q_2, \cdots, q_t$ be distinct prime divisors of $n(n+1)$ and $(n+2)(n+3)$ respectively. We may assume that

$$p_1 < p_2 < \cdots < p_k,\ q_1 < q_2 < \cdots < q_t. \tag{4.7}$$

Since gcd $(n, n+1)$=gcd $(n+2, n+3)=1$, by Theorem 1.1, we get

$$\begin{aligned} ZW(n).\ ZW(n+1) &= p_1 p_2 \cdots p_k \\ ZW(n+2).\ ZW(n+3) &= q_1 q_2 \cdots q_t \end{aligned} \tag{4.8}$$

Substitute (4.8) into (4.3), we obtain

$$p_1 p_2 \cdots p_k = q_1 q_2 \cdots q_t. \tag{4.9}$$

By (4.7) and (4.9), we get $k=t$ and

$$p_i = q_i,\ i=1, 2, \cdots, k. \tag{4.10}$$

Since gcd $(n+1, n+2)=1$, if $2|n$ and p_j $(1 \leqslant j \leqslant k)$ is a prime divisor of $n+1$, then from (4.10) we see that p_j is an odd prime with $p_j|n+3$.

Since gcd $(n+1, n+3)=1$ if $2| n$, it is impossible.

Similarly, if $2 \mid n$ and q_j $(i \leqslant j \leqslant k)$ is a prime divisor of $n+2$, then q_j is an odd prime with $q_j | n$. However, since $(n, n+2)=1$ if $2 \mid n$, it is impossible. Thus, (4.3) has no solutions n. The theorem is proved.

Theorem 4.4. The equation (4.4) has infinitely many solutions (m, n, k). Moreover, every solution (m, n, k) of (4.4) can be expressed as

$$m=p_1p_2\cdots p_r,\ n=t,\ k=1, \tag{4.11}$$

where $p_1, p_2, \cdots, p_r$ are distinct primes, t is a positive integer with gcd $(m, t)=1$.

Proof. Let (m, n, k) be a solution of (4.4). Further let $d=\gcd(m, n)$. By Theorem 1.1, we get from (4.4) that

$$ZW(mn)=ZW\left(\frac{m}{d}.n\right)=ZW\left(\frac{m}{d}\right).ZW(n)=m^kZW(n). \tag{4.12}$$

Since $ZW(n)\neq 0$, we obtain from (4.12) that

$$ZW\left(\frac{m}{d}\right)=m^k. \tag{4.13}$$

Furthermore, since $m \geqslant ZW(m)$, we see from (4.13) that $k=d=1$ and $m=p_1p_2\cdots p_r$, where $p_1, p_2, \cdots, p_r$ are distinct primes. Thus, the solution (m, n, k) can be expressed as (4.11). The theorem is proved.

Theorem 4.5. The equation (4.5) has infinitely many solutions (n, k). Moreover, every solution (n, k) of (4.5) can be expressed as

$$n=2^r,\ k=2,\ r\in \mathbf{N}. \tag{4.14}$$

Proof. Let (n, k) be a solution of (4.5). Further let $d=\gcd(n, k)$. By Theorem 1.1, we get from (4.5) that

$$ZW(nk)=kZW\left(n.\frac{k}{d}\right)=kZW(n).ZW\left(\frac{k}{d}\right)=(ZW(n))^k. \tag{4.15}$$

Since $ZW(n)\neq 0$ and $k>1$, by (4.15), we obtain

$$kZW\left(\frac{k}{d}\right)=(ZW(n))^{k-1}. \tag{4.16}$$

Since $n>1$, we find from (4.16) that k and n have the same prime divisors.

Let $p_1, p_2, \cdots, p_t$ be distinct prime divisors of n. Then we have $ZW(n)=p_1p_2\cdots p_t$. Since $ZW(k/d)\leqslant k$, we get from (4.16) that

$$k^2 \geq kZW\left(\frac{k}{d}\right)=(ZW(n))^{k-1}=(p_1p_1\ldots p_t)^{k-1}. \tag{4.17}$$

Since $k>1$, by (4.17), we obtain $t=1$ and either

$$k=3, p_1=3, \tag{4.18}$$

or

$$k=2, p_1=2. \tag{4.19}$$

Recall that k and n have the same prime divisors. If (4.18) holds, then $ZW(k/d)=ZW(1)=1$ and (4,16) is impossible. If (4.19) holds, then the solution (n, k) can be expressed as (4.14). Thus, the Theorem is proved.

Theorem 4.6. The equation (4.6) has no solutions (n, k).

Proof. Let (n, k) be a solution of (4.6). Further let $m=ZW(n)$, and let $p_1, p_2, \cdots, p_t$ be distinct prime divisors of n. By Theorem 1.1, we have

$$n=p_1^{a_1}p_2^{a_2}\ldots p_t^{a_t}, ZW(n)=p_1p_2\ldots p_t, \tag{4.20}$$

where $a_1, a_2, \cdots, a_t$ are positive integers. Substitute (4.20) into (4.6), we get

$$1+p_1p_2\ldots p_t+\ldots+(p_1p_2\ldots p_t)^{k-1}=p_1^{a_1-1}p_2^{a_2-1}\ldots p_t^{a_t-1}. \tag{4.21}$$

Since gcd $(1, p_1p_2\cdots p_t)=1$, we find from (4.21) that $a_1=a_2=\cdots=a_t=1$. It

implies that k=1, a contradiction. Thus, (4.6) has no solutions (n, k). The theorem is proved.

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THE EQUATION $S(1.2)+S(2.3)+\cdots+S(n(n+1))=S(n(n+1)(n+2)/3)$

Maohua Le

Abstract. For any positive integer a, let $S(a)$ be the Smarandache function of a. In this paper we prove that the title equation has only the solution $n=1$.

Key words: Smarandache function, diophantine equation

Let **N** be the set of all positive integers. For any positive integer a, let $S(a)$ be the Smarandache function of a. Recently, Bencze [1] proposed the following problem:

Problem Solve the equation

$$(1)\quad S(1\cdot 2)+S(2\cdot 3)+\cdots+S(n(n+1))=S\left(\frac{1}{3}n(n+1)(n+2)\right),\ n\in\mathbf{N}.$$

In this paper we completely solve the above-mentioned problem as follows.

Theorem The equation (1) has only the solution $n=1$.

Proof By the definition of the Smarandache function (see [2]), we have $S(1)=1$, $S(2)=2$ and

$$(2)\qquad S(a)\geqslant 3,\ a\geqslant 3.$$

Since $S(1.2)=S(1.2.3/3)=S(2)$, the equation (1) has a solution $n=1$.

Let n be a solution of (1) with $n>1$. Then, by (2), we get

$$(3)\quad S(1\cdot 2)+S(2\cdot 3)+\cdots+S(n(n+1))\geq 2+3(n-1)=3n-1.$$

Therefore, by (1) and (3), we obtain

(4) $$S\left(\frac{1}{3}n(n+1)(n+2)\right)\geq 3n-1.$$

On the other hand, since $(n+2)!=1.2\cdots n(n+1)(n+2)$, we get

(5) $$\frac{1}{3}n(n+1)(n+2)\,|\,(n+2)!.$$

We see from (5) that

(6) $$S\left(\frac{1}{3}n(n+1)(n+2)\right)\leq n+2.$$

The combination of (4) and (6) yields

(7) $$n+2\geq 3n-1,$$

whence we get $n\leqslant 3/2<2$. Since $n\geqslant 2$, it is impossible. Thus, (1) has no solutions n with $n>1$. The theorem is proved.

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SOME CONJECTURES ON PRIMES (I)

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Abstract. For any complex number x, let $\exp(x)=e^x$. For any positive integer n, let p_n be the nth prime. In this paper we prove that $\exp\left(\sqrt{(n+1)/p_{n+1}}\right)/\exp\left(\sqrt{p_n/n}\right)<\exp\left(\sqrt{3/5}\right)/\exp\left(\sqrt{3/2}\right)$.

Key words: prime, inequality.

For any complex number x, let $\exp(x)=e^x$. For any positive integer n, let p_n be the nth prime. Recently, Russo [2] proposed the following conjecture:

Conjecture For any positive integer n,

$$\frac{\exp\left(\sqrt{\frac{n+1}{p_{n+1}}}\right)}{\exp\left(\sqrt{\frac{p_n}{n}}\right)}<\frac{\exp\left(\sqrt{\frac{3}{5}}\right)}{\exp\left(\sqrt{\frac{3}{2}}\right)}. \tag{1}$$

In [2], Russo verified (1) for $p_n\leqslant 10^7$. In this paper we completely solve the above-mentioned conjecture as follows.

Theorem For any positive integer n, the inequality (1) holds.

Proof We may assume that $p_n>10^7$. Then we have $n>1000$. It is a well known fact that

$$p_n>n\log n, \tag{2}$$

for any positive integer n (see [1]). By (2), we get

(3) $$\exp\left(\sqrt{\frac{p_n}{n}}\right)>\exp\left(\sqrt{\log n}\right)>\exp\left(\sqrt{\log 1000}\right)>\exp(2.6).$$

On the other hand, since $p_{n+1}>n+1$, we get

(4) $$\exp\left(\sqrt{\frac{3}{2}}-\sqrt{\frac{3}{5}}+\sqrt{\frac{n+1}{p_{n+1}}}\right)<\exp\left(\sqrt{\frac{3}{2}}-\sqrt{\frac{3}{5}}+1\right)<\exp(1.5).$$

The combination of (3) and (4) yields

(5) $$\exp\left(\sqrt{\frac{p_n}{n}}\right)>\exp\left(\sqrt{\frac{3}{2}}-\sqrt{\frac{3}{5}}+\sqrt{\frac{n+1}{p_{n+1}}}\right).$$

Thus, by (5), we get (1) immediately. The theorem is proved.

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SOME CONJECTURES ON PRIMES (II)

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Abstract. For any positive integer n, let p_n be the nth prime. In this paper we give a counter-example for the inequality

$$\exp\left(\sqrt{(n+1)/p_{n+1}}\right)/\exp\left(\sqrt{p_n/n}\right)<\exp\left(\sqrt{3/5}\right)/\exp\left(\sqrt{3/2}\right).$$

Key words: prime, inequality

For any positive integer n, let p_n be the nth prime. Recently, Russo [3] proposed the following conjecture:

Conjecture For any positive integer n,

$$|p_n\cdot(n+1)-n\cdot p_{n+1}|<\frac{1}{2}(n+1)^{9/50}. \tag{1}$$

In [3], Russo verified the equality (1) holds for $p_n\leqslant 10^7$. However, we shall show that (1) is false for some n.

Let p_n and p_{n+1} be twin primes. Then we have

$$p_{n+1}=p_n+2. \tag{2}$$

If (1) holds, then from (2) we get

$$|p_n-2n|<\frac{1}{2}(n+1)^{9/50}. \tag{3}$$

It is a well known fact that

$$p_n>n\log n \tag{4}$$

for any positive integer n (see [2]). Therefore, by (3) and (4), we obtain

(5) $$n(\log n-2)<\frac{1}{2}(n+1)^{9/50},n>6.$$

By [1], p_n=297.2^{546}-1 and p_n+1=297.2^{546}+1 are twin primes. Then we have $n>10^{10}$. Therefore, (5) is impossible. Thus, the inequality (1) is false for some n.

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SOME CONJECTURES ON PRIMES (III)

Maohua Le

Abstract. For any positive integer n, let p_n be the nth prime. In this paper we prove that the equality

$$\left(\sqrt{p_n}-\log p_{n+1}\right)/\left(\sqrt{p_{n+1}}-\log p_n\right)\geq\left(\sqrt{3}-\log 5\right)/\left(\sqrt{5}-\log 3\right)\text{for any n.}$$

Key words: prime, inequality.

For any positive integer n, let p_n be the nth prime. Recently, Russo [2] proposed the following conjecture:

Conjecture For any positive integer n,

$$\frac{\sqrt{p_n}-\log p_{n+1}}{\sqrt{p_{n+1}}-\log p_n}\geq\frac{\sqrt{3}-\log 5}{\sqrt{5}-\log 3}. \tag{1}$$

In [2], Russo verified the equality (1) holds for $p_n\leqslant 10^7$. In this paper, we completely solve the above-mentioned problem as follows.

Theorem For any positive integer n, the equality (1) holds.

Proof We may assume that $p_n>10^7$. Since

$$\frac{\sqrt{3}-\log 5}{\sqrt{5}-\log 3}<0.11, \tag{2}$$

if (1) is false, then from (2) we get

$$\sqrt{p_n}<\log p_{n+1}+0.11\sqrt{p_{n+1}}. \tag{3}$$

It is a well known fact that

(4) $$p_{n+1}<2p_n$$

for any positive integer n (see [1, Theorem 245]). Substitute (4) into (3), we obtain

(5) $$0.84\sqrt{p_n}<\log(2p_n).$$

However, (5) is impossible for $p_n>10^7$. Thus, the theorem is proved.

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SMARANDACHE COSETS

W. B. Vasantha Kandasamy

Abstract

This paper aims to study the Smarandache cosets and derive some interesting results about them. We prove the classical Lagranges theorem for Smarandache semigroup is not true and that there does not exist a one-to-one correspondence between any two right cosets. We also show that the classical theorems cannot be extended to all Smarandache semigroups. This leads to the definition of Smarandache Lagrange semigroup, Smarandache p Sylow subgroup and Smarandache Cauchy elements. Further if we restrict ourselves to the subgroup of the Smarandache semigroup all results would follow trivially hence the Smarandache coset would become a trivial definition.

Keywords:

Smarandache cosets, Smarandache Lagrange semigroups, Smarandache p-Sylow subgroups, Smarandache Cauchy element, Smarandache Normal subgroups and Smarandache quotient groups.

Definition [2]: The *Smarandache semigroup* is defined to be a semigroup A such that a proper subset of A is a group (with respect to the same induced operation).

Definition 1. Let A be a Smarandache semigroup. A is said to be a *commutative Smarandache semigroup* if the proper subset of A that is a group is commutative.
If A is a commutative semigroup and if A is a Smarandache semigroup then A is obviously a commutative Smarandache semigroup.

Definition 2. Let A be a Smarandache semigroup. $H \subseteq A$ be a group under the same operations of A. For any $a \in A$ the *Smarandache right coset* is $Ha = \{ha / h \in H\}$. Ha is called the Smarandache right coset of H in A. Similarly left coset of H in A can be defined.

Example 1: Let $Z_{12} = \{0, 1, 2, ..., 11\}$ be the Smarandache semigroup under multiplication modulo 12. Clearly Z_{12} is a commutative Smarandache semigroup. Let $A = \{3,9\}$ be a subgroup of Z_{12} under multiplication. $9^2 = 9 \pmod{12}$ acts as identity with respect to multiplication. For $4 \in Z_{12}$ the right (left) coset of A in Z_{12} is $4A = \{0\}$. For $1 \in Z_{12}$ the right (left) coset of A in Z_{12}

is $1A = \{3, 9\}$. Hence we see the number of elements in nA is not the same for each $n \in Z_{12}$.

Example 2. $Z_9 = \{0, 1, 2, ..., 8\}$ be the commutative Smarandache semigroup under multiplication modulo 9. $A = \{1, 8\}$ and $A_1 = \{2, 4, 1, 5, 7, 8\}$ are the subgroups of Z_9. Clearly order of A does not divide 9. Also order of A_1 does not divide 9.

Example 3. Let S denote the set of all mappings from a 3-element set to itself. Clearly number of elements in S is 27. S is a semigroup under the composition of maps.

Now S contains S_3 the symmetric group of permutations of degree 3. The order of S_3 is 6. Clearly 6 does not divide order of S.

Thus we see from the above examples that the classical Lagrange theorem for groups do not hold good for Smarandache semigroups. It is important to mention here that the classical Cayley theorem for groups could be extended to the case of Smarandache semigroups. This result is proved in [3]. For more details please refer [3]. Thus:

Definition 3. Let S be a finite Smarandache semigroup. If the order of every subgroup of S divides the order of S then we say S is a *Smarandache Lagrange semigroup.*

Example 4. Let $Z_4 = \{0, 1, 2, 3\}$ be the semigroup under multiplication. $A = \{1, 3\}$ is the only subgroup of Z_4. Clearly $|A|/4$. Hence Z_4 is a Smarandache Lagranges semigroup.

But we see most of the Smarandache semigroups are not Smarandache Lagrange semigroup. So one has:

Definition 4. Let S be a finite Smarandache semigroup. If there exists at least one group, i.e. a proper subset having the same operations in S, whose order divides the order of S, then we say that S is a *weakly Smarandache Lagrange semigroup.*

Theorem 5. Every Smarandache Lagrange semigroup is a weakly Smarandache Lagrange semigroup and not conversely.

Proof: By the very definition 3 and 4 we see that every Smarandache Lagrange semigroup is a weakly Smarandache Lagrange semigroup.

To prove the converse is not true consider the Smarandache semigroup given in Example 3. 6 does not divide 27 so S is not a Smarandache Lagrange

semigroup but S contains subgroup say $\left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix},\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix},\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}\right\}$ of order 3. Clearly 3 divides 27. Thus S is a weakly Smarandache Lagrange semigroup.

Thus the class of Smarandache Lagrange semigroup is strictly contained in the class of weakly Smarandache Lagrange semigroup.

Theorem 6. Let S = {1, 2, ..., n}, $n \geq 3$, be the set with n natural elements, S(n) the semigroup of mappings of the set S to itself. Clearly S(n) is a semigroup under the composition of mapping. S(n) is a weakly Smarandache Lagrange semigroup.

Proof: Clearly order of $S(n) = n^n$. S_n the symmetric group of order n!. Given $n \geq 3$, n! does not divide n^n for

$$\frac{n^n}{n!} = \frac{\overbrace{n \times \cdots \times n}^{n \text{ times}}}{1.2.3.4.\cdots.n-1.n} = \frac{\overbrace{n \times \cdots \times n}^{n-1 \text{ times}}}{1.2.\cdots.n-1}$$

Now since (n -1, n) = 1, that is n - 1 and n are relatively prime. We see n! does not divide n^n. Hence the class of Smarandache semigroups S(n), $n \geq 3$, are weakly Smarandache Lagrange semigroup.

Corollary. S(n) , n = 2, is a Smarandache Lagrange semigroup.

Proof: Let n = 2. Then $S(n) = \left\{\begin{pmatrix}1&2\\1&1\end{pmatrix},\begin{pmatrix}1&2\\2&2\end{pmatrix},\begin{pmatrix}1&2\\2&1\end{pmatrix},\begin{pmatrix}1&2\\1&2\end{pmatrix}\right\}$, |S(n)| = 4. $S_2 = \left\{\begin{pmatrix}1&2\\1&2\end{pmatrix},\begin{pmatrix}1&2\\2&1\end{pmatrix}\right\}$ is the symmetric group of degree 2 and $|S_2|$ divides 4^2. Hence the claim.

Now the natural question would be: does there exist a Smarandache semigroup, which are not a Smarandache Lagrange semigroup and weakly Smarandache Lagrange semigroup? The answer is yes. The Smarandache semigroup Z_9 = {0, 1, 2 , ..., 8} under multiplication given in example 2 does not have subgroups which divides 9, hence the claim.

Now to consider the converse of the classical Lagrange theorem we see that there is no relation between the divisor of the order of the Smarandache semigroup S and the order of the subgroup S contains. The example is quite interesting.

Example 5: Let $Z_{10} = \{0, 1, 2, \ldots, 9\}$ be the semigroup of order 10. Clearly Z_{10} is a Smarandache semigroup. The subgroups of Z_{10} are $A_1 = \{1, 9\}$, $A_2 = \{2, 4, 6, 8\}$ and $A_3 = \{1, 3, 7, 9\}$, $A_4 = \{4, 6\}$. Thus 4 does not divide 10, which contradicts Lagrange's theorem (that the order of a subgroup divides the order of the group) in the case of Smarandache semigroup. Also Z_{10} has subgroups of order 5 leading to a contradiction of the classical Sylow theorem (which states that if p^α divides the order of the group G then G has a subgroup of order p^α) again in the case of Smarandache semigroup. This forces us to define Smarandache p-Sylow subgroups of the Smarandache semigroup.

Definition 7. Let S be a finite Smarandache semigroup. Let p be a prime such that p divides the order of S. If there exists a subgroup A in S of order p or p^t ($t > 1$) we say that S has a *Smarandache p-Sylow subgroup.*

Note. It is important to see that p^t needs not to divide the order of S, that is evident from Example 5, but p should divide the order of S.

Example 6. Let $Z_{16} = \{0, 1, 2, \ldots, 15\}$ be the Smarandache semigroup of order $16 = 2^4$. The subgroups of Z_{16} are $A_1 = \{1, 15\}$, $A_2 = \{1, 3, 9, 11\}$, $A_3 = \{1, 5, 9, 13\}$, and $A_4 = \{1, 3, 5, 7, 9, 11, 13, 15\}$ of order 2, 4, and 8 respectively. Clearly the subgroup A_4 is the Smarandache 2-Sylow subgroup of Z_{16}.

The Sylow classical theorems are left as open problems in case of Smarandache p-Sylow subgroups of a Smarandache semigroup.

Problem 1. Let S be a finite Smarandache semigroup. If $p/|S|$ and S has Smarandache p-Sylow subgroup. Are these Smarandache p-Sylow subgroups conjugate to each other?

Problem 2. Let S be a finite Smarandache semigroup. If p divides order of S and S has Smarandache p-Sylow subgroups. How many Smarandache p-Sylow subgroups exist in S?

Let S be a finite Smarandache semigroup of order n. Let $a \in S$ now for some $r > 1$, if $a^r = 1$ then in general r does not divide n.

Example 7. Let $S = \{1, 2, 3, 4, 5\}$ be the set with 5 elements S (5) be the semigroup of mappings of S to itself. S(5) is a Smarandache semigroup for S(5) contains S_5 the permutation group of degree 5. Clearly $|S(5)| = 5^5$. Now $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 5 \end{pmatrix} \in S(5)$. Clearly $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 5 \end{pmatrix}^4$ = identity element of S(5), but 4 does not divide $|S(5)| = 5^5$. Thus we define Smarandache Cauchy element.

Definition 8. Let S be a finite Smarandache semigroup. An element $a \in A$, $A \subset S$, A the subgroup of S, is said to be a *Smarandache Cauchy element* of S if $a^r = 1$ ($r \geq 1$), 1 unit element of A, and r divides the order of S; otherwise a is not a Smarandache Cauchy element.

Problem 3. Can we find conditions on the Smarandache semigroup S so that every element in S is a Smarandache Cauchy element of S?

Problem 4. Let Z_n be the Smarandache semigroup under usual multiplication modulo n. Is every element in every subgroup of Z_n is a Cauchy element of Z_n? (n is not a prime.)

Remark: $Z_n = \{0, 1, 2, ..., n-1\}$ is a Smarandache semigroup under multiplication. Clearly every x in Z_n is such that $x^r = 1$ ($r > 1$), but we do not whether every element in every subgroup will satisfy this condition. This is because the subgroups may not have $1 \in Z_n$ as the identity element.

Definition 9. Let S be a finite Smarandache semigroup, if every element in every subgroup of S is a Smarandache Cauchy element; then we say S is a *Smarandache Cauchy semigroup.*

Theorem 10. Let S(n) be the Smarandache semigroup for some positive integer n. S(n) is not a Smarandache Cauchy semigroup.

Proof: Clearly S_n is a subgroup of S (n). We know $|S(n)| = n^n$ and $|S_n| = \underline{n}$. But S_n contains elements x of order (n-1), and (n-1) does not divide n^n. So S (n) is not a Smarandache Cauchy semigroup.

Thus we see the concept of the classical theorem on Cauchy group cannot be extended to finite Smarandache semigroups.

Theorem 11. There does not exist in general a one-to-one correspondence between any two Smarandache right cosets of A in a Smarandache semigroup S.

Proof: We prove this by the following example. Let $S = Z_{10} = \{0, 1, 2, ...,9\}$. $A = \{1, 9\}$ is a subgroup of S. $A_2 = \{2, 4, 6, 8\}$ is a subgroup of S. $3A = \{3,7\}$ and $5A = \{5\}$. Also $5A_2 = \{0\}$ and $3A_2 = A_2$. So there is no one-to-one correspondence between Smarandache cosets in a Smarandache semigroup.

Theorem 12. The Smarandache right cosets of A in a Smarandache semigroup S does not in general partition S into either equivalence classes of same order or does not partition S at all.

Proof: Consider Z_{10} given in the proof of Theorem 12. Now for A = {1, 9} the subgroup of Z_{10} that is the coset division of Z_{10} by A are {0}, {5}, {1, 9}, {2,8}, {3, 7} and {4, 6}. So A partitions S as cosets the Smarandache semigroup into equivalence classes but of different length. But for A_2 = {2,4,6,8} is a subgroup of Z_{10}. 6 acts as the identity in A_2. Now the coset of division of Z_{10} by A_2 is {2,4,6,8} and {0} only. Hence this subsets do not partition Z_{10}.

Problem 5. Does there exist any Smarandache semigroup S such that there is one-to-one correspondence between cosets of A in S?

Now we proceed to define Smarandache double cosets of a Smarandache semigroup S.

Definition 13. Let S be a Smarandache semigroup. $A \subset S$ and $B \subset S$ be any two proper subgroups of S. For $x \in S$ define $AxB = \{axb / a \in A, b \in B\}$. AxB is called a *Smarandache double coset* of A and B in S.

Example 8: Let $Z_{10} = \{0, 1, 2, ..., 9\}$. A = {1, 9} and B = {2, 4, 6, 8} be subgroups of the commutative Smarandache semigroup of order 10. Take x = 5 then AxB = {0}. Take x = 3 then AxB = {2, 4, 6, 8}. For x = 7, AxB = {2, 4, 6, 8}. Thus Z_{10} is not divided into equivalence classes by Smarandache double cosets hence we have the following theorem.

Theorem 14. Smarandache double coset relation on Smarandache semigroup S is not an equivalence relation on S.

Definition 15. Let S be a Smarandache semigroup. Let A be a proper subset of S that is a group under the operations of S. We say A is a *Smarandache normal subgroup* of the Smarandache semigroup S if $xA \subseteq A$ and $Ax \subseteq A$ or xA = {0} and Ax = {0} for all $x \in S$ if 0 is an element in S.

Note. As in case of normal subgroups we cannot define $xAx^{-1} = A$ for every $x \in S$, x^{-1} may not exist. Secondly if we restrict our study only to the subgroup A it has nothing to do with Smarandache semigroup for every result is true in A as A is a group.

Example 9. Let $Z_{10} = \{0, 1, 2, ..., 9\}$ be a Smarandache semigroup of order 10. A = {2,4,6,8} is a subgroup of Z_{10} which is a Smarandache normal subgroup of Z_{10}. It is interesting to note that that order of the normal subgroup of a Smarandache semigroup needs in general not to divide the order of the Smarandache semigroup. So if we try to define a Smarandache quotient group it will not be in general a group.

Definition 16. Let S be a Smarandache semigroup and A a Smarandache normal subgroup of S. The *Smarandache quotient group* of the Smarandache semigroup S is $\frac{S}{A} = \{Ax / x \in S\}$.

Note. $\frac{S}{A}$ in general is not a group, it is only a semigroup. Further, as in classical group theory, the number of elements in $\frac{S}{A}$ or in A or in S look in general not to be related. Earlier example of Z_{10}, $|Z_{10}| = 10$, $|A| = 4$ and $\left|\frac{Z_{10}}{A}\right| = 2$ proves this note.

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SMARANDACHE LOOPS

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Abstract

In this paper we study the notion of Smarandache loops. We obtain some interesting results about them. The notion of Smarandache semigroups homomorphism is studied as well in this paper. Using the definition of homomorphism of Smarandache semigroups we give the classical theorem of Cayley for Smarandache semigroups. We also analyze the Smarandache loop homomorphism. We pose the problem of finding a Cayley theorem for Smarandache loops. Finally we show that all Smarandache loops $L_n(m)$ for $n > 3$, n odd, varying n and appropriate m have isomorphic subgroups.

Keywords:

Loops, Bruck Loop, Bol loop, Moufang loop, Smarandache loop, power associative loop, right or left alternative loop, Smarandache semigroup homomorphism, Smarandache loop homomorphism.

Definition [1, Bruck]:

A non-empty set L is said to form a loop if on L is defined a binary operation called product denoted by '$\bullet$' such that

1. For a, b $\in$ L, we have a $\bullet$ b $\in$ L
2. There exists an element e $\in$ L such that a $\bullet$ e = e $\bullet$ a = a for all a $\in$ L (e called identity element of L)
3. For every ordered pair (a,b) $\in$ L $\times$ L there exists a unique pair (x, y) $\in$ L $\times$ L such that a $\bullet$ x = b and y $\bullet$ a = b.

By a loop, we mean only a finite loop and the operation '$\bullet$' need not always be associative for a loop. A loop is said to be a Moufang Loop if it satisfies any one of the following identity.

1. (xy)(zx) = (x(yz))x
2. ((xy)z)y = x(y(zy))
3. x(y(xz)) = ((xy)x)z

for all x, y, z $\in$ L.

A loop L is said to be Bruck Loop if $x(yx)z = x(y(xz))$ and $(xy)^{-1} = x^{-1}y^{-1}$ for all x, y, z $\in$ L. L is a Bol Loop if $((xy)z)y = x((yz)y)$ for all x, y, z $\in$ L. L is a right alternative loop if $(xy)y = x(yy)$ for all x, y $\in$ L and left alternative if $(xx)y = x(xy)$. L is said to be an alternative loop if it is both a right and a left alternative loop. A loop L is said to be power associative if every element generates a subgroup. L is said to be di-associative if every 2 elements of L generates a subgroup. Let $L_n(m) = (e, 1, 2, 3, \ldots, n)$ be a set where $n > 3$, n is odd and m is a positive integer such that $(m,n) = 1$ and $(m-1, n) = 1$ with $m < n$. Define on $L_n(m)$ a binary operation '$\bullet$' as follows .

1. $e \bullet i = i \bullet e = i$ for all $i \in L_n(m)$
2. $i \bullet i =$ for all $i \in L_n(m)$
3. $i \bullet j = t$ where $t = (mj-(m-1)\ i) \pmod n$ for all $i \bullet j \in L_n(m)$ $i \neq j$, $i \neq e$ and $j \neq e$. $L_n(m)$ is a loop.

We call this a new class of loops.

For more about loops and its properties please refer to [1] , [5] , [6] , [7] , [8] , [9] , [10], [11], [12] and [13] .

Definition 1:

The *Smarandache Loop* is defined to be a loop L such that a proper subset A of L is a subgroup (with respect to the same induced operation). That is $\phi \neq A \subset L$.

Example 1

Let L be a loop given by the following table

$\bullet$	e	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_2	e	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_5	a_1	e	a_5	a_2	a_6	a_3	a_7	a_4
e	a_2	a_5	e	a_6	a_3	a_7	a_4	a_1
a_6	a_3	a_2	a_6	e	a_7	a_4	a_1	a_5
a_3	a_4	a_6	a_3	a_7	e	a_1	a_5	a_2
a_7	a_5	a_3	a_7	a_4	a_1	e	a_2	a_6
a_4	a_6	a_7	a_4	a_1	a_5	a_2	e	a_3
a_1	a_7	a_4	a_1	a_5	a_2	a_6	a_3	e

L is a Smarandache loop. For the pair (e, a2) is a subgroup of L.

Theorem 2

Every power associative loop is a Smarandache loop.

Proof

By definition of a power associative loop every element in L generates a subgroup in L. Hence the proof.

Theorem 3

Every di-associative loop is a Smarandache loop.

Proof

Since in a di-associative loop L every two elements of L generate a subgroup in L. So every di - associative loop has subgroups, hence L is a Smarandache loop.

Theorem 4

Every loop $L_n(m)$ for $n > 3$, n an odd integer. $(n,m) = (n, m-1) = 1$ with $m < n$ is a Smarandache loop.

Proof

Since $L_n(m)$ is power associative we have for every a in $L_n(m)$ is such that $a^2 = e$, $\{a,e\}$ forms a subgroup for every a in $L_n(m)$. Hence the claim. Thus it is interesting to note that for every odd integer n there exists a class of Smarandache loops of order n+1. For a given $n > 3$, n odd we can have more than one integer m, $m < n$ such that $(m,n) = (m-1, n) = 1$. For instance when $n = 5$ we have only 3 Smarandache loops given by $L_5(2)$, $L_5(3)$ and $L_5(4)$.

Definition 5

The *Smarandache Bol loop* is defined to be a loop L such that a proper subset A of L is a Bol loop (with respect to the same induced operation). That is $\phi \neq A \subset S$.

Note 1 - Similarly is defined *Smarandache Bruck loop, Smarandache Moufang loop* and *Smarandache right (left) alternative loop.*

Note 2- In definition 5 we insist that A should be a subloop of L and not a subgroup of L. For every subgroup is a subloop but a subloop in general is not a subgroup. Further every subgroup will certainly be a Moufang loop, Bol loop, Bruck loop and right(left) alternative loop, since in a group the operation is associative. Hence only we make the definition little rigid so that automatically we will not have all Smarandache loops to be Smarandache Bol loop, Smarandache Bruck loop, Smarandache Moufang loop and Smarandache right (left) alternative loop.

Theorem 6

Every Bol loop is a Smarandache Bol loop but every Smarandache Bol loop is not a Bol loop.

Proof

Clearly every Bol loop is a Smarandache Bol loop as every subloop of a Bol loop is a Bol loop. But a Smarandache Bol loop L is one which has a proper subset A which is a Bol loop. Hence L need not in general be a Bol loop.

Definition 7

Let S and S′ be two Smarandache semigroups. A map ϕ from S to S′ is said to be a *Smarandache semigroup homomorphism* if ϕ restricted to a subgroup $A \subset S$ and $A' \subset S'$ is a group homomorphism, that is $\phi : A \subset S \to A' \subset S'$ is a group homomorphism. The Smarandache semigroup homomorphism is an isomorphism if $\phi : A \to A'$ is one to one and onto.
Similarly, one can define *Smarandache semigroup automorphism* on S.

Theorem 8

Let N be any set finite or infinite. S(N) denote the set of all mappings of N to itself. S(N) is a semigroup under the composition of mappings. S(N), for every N, is a Smarandache semigroup.

Proof

S(N) is a semigroup under the composition of mappings. Now let A(N) denote the set of all one to one mappings of N. Clearly $\phi \neq A(N) \subset S(N)$ and A(N) is a subgroup of S(N) under the operation of composition of mappings, that is A(N) is the permutation group of degree N. Hence S(N) is a Smarandache semigroup for all N> 1.

Example 2

Let S = {set of all maps from the set (1, 2, 3, 4) to itself} and S′ = {set of all map from the set (1, 2, 3, 4, 5, 6) to itself}. Clearly S and S′ are Smarandache semigroups. For $A = S_4$ is the permutation subgroup of S and $A' = S_6$ is also the permutation subgroup of S′. Define the map $\phi : S \to S'$, $\phi(A) = B'$ = {set of all permutations of (1, 2, 3, 4) keeping the positions of 5 and 6 fixed} $\subseteq A'$. Clearly ϕ is a Smarandache semigroup homomorphism.

From the definition of Smarandache semigroup homomorphism one can give the modified form of the classical Cayley's theorem for groups to Smarandache semigroups.

Theorem 9 (Cayley's Theorem for Smarandache semigroups)

Every Smarandache semigroup is isomorphic to a Smarandache semigroup of mappings of a set N to itself, for some appropriate set N.

Proof

Let S be a Smarandache semigroup. That is there exists a set A, which is a proper subset of S, such that A is a group (under the operations of S), that is $\phi \neq A \subset S$. Now let N be any set and S(N) denotes the set of all mappings from N to N. Clearly S(N) is a Smarandache semigroup. Now using the classical Cayley' s theorem for groups we can always have an isomorphism from A to a subgroup of S(N) for an appropriate N. Hence the theorem.

Thus by defining the notion of Smarandache semigroups one is able to extend the classical theorem of Cayley. Now we are interested to finding the appropriate formulation of Cayley's theorem for loops.

It is important to mention here that loops do not satisfy Cayley's theorem but for Smarandache loops the notion of Cayley's theorem unlike Smarandache semigroups is an open problem.

Definition 10

Let L and L′ be two Smarandache loops with A and A′ its subgroups respectively. A map ϕ from L to L′ is called *Smarandache loop homomorphism* if ϕ restricted to A is mapped to a subgroup A′ of L′, that is $\phi : A \to A'$ is a group homomorphism. The concept of *Smarandache loop homomorphism and automorphism* are defined in a similar way.

Problem 1 Prove or disprove that every Smarandache loop L is isomorphic with a Smarandache Loop L′ or equivalently

Problem 2 Can a loop L′ be constructed having a proper appropriate subset A′ of L′ such that A′ is a desired subgroup $\phi \neq A' \subset L'$?

Problem 3 Characterize all Smarandache loops which have isomorphic subgroups ?

Example 3

Let $L_5(3)$ be a Smarandache loop given by the following table

•	e	1	2	3	4	5
e	e	1	2	3	4	5
1	1	e	4	2	5	3
2	2	4	e	5	3	1
3	3	2	5	e	1	4
4	4	5	3	1	e	2
5	5	3	1	4	2	e

and $L_7(3)$ is another Smarandache loop given by the following table

•	e	1	2	3	4	5	6	7
e	e	1	2	3	4	5	6	7
1	1	e	4	7	3	6	2	5
2	2	6	e	5	1	4	7	3
3	3	4	7	e	6	2	5	1
4	4	2	5	1	e	7	3	6
5	5	7	3	6	2	e	1	4
6	6	5	1	4	7	3	e	2
7	7	3	6	2	5	1	4	e

These two loops have isomorphic subgroup, for $L_7(3)$ and $L_5(3)$ have subgroups of order 2.

Theorem 11

All Smarandache loops $L_n(m)$, where $n > 3$, n odd, for varying n and appropriate m, have isomorphic subgroups.

Proof

All Smarandache loops $L_n(m)$ have subgroups of order 2. Hence they have isomorphic subgroups.

Note- *This does not mean $L_n(m)$ cannot have subgroups of order other than two. the main concern is that all loops $L_n(m)$ have subgroups of order 2.*

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A Discrete Model for Histogram Shaping

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Abstract

The aim of this article is to present a discrete model for histogram shaping. This is an important image transformation with several practical applications. The model that is proposed is based on a generalization of the inferior part function. Finally, an algorithm based on this model is developed.

Key Words: histogram, histogram shaping, discrete random variable.

1 Introduction

Histogram equalization or histogram flattening is one of the most important nonlinear point operations. This transformation aims to distribute uniformly the gray levels of the input image such that the histogram of the output image is flat. Histogram equalization has been studied for many years (see [1], [3], [4]) and many practical applications have been proposed so far. A direct generalization of this transformation is represented by histogram shaping or histogram specification (see [1], [3]). The idea of histogram shaping is to transform the input image into one which has histogram of a specific shape. Obviously, when the output shape is flat, histogram equalization is obtained. Both histogram equalization and histogram shaping have become classical image transformations, therefore it has been quite difficult to find the initial reference source. One of the earliest references about is [2].

The mathematical model of histogram equalization and shaping is based on stochastic approach. Let us consider that the input digital image is $f = (f_{i,j} : \ i = 1,2,...,n; \ j = 1,2,...,m)$ where

$$1 \leq f_{i,j} \leq G$$

represents the gray value of pixel (i,j). The probability or frequency of gray level $k \in 1,...,G$ is defined by

$$p_f(k) = \frac{\#\{(i,j) : \ f(i,j) = k\}}{m \cdot n}, \ k = 1,...,G, \tag{1}$$

where $\#\{(i,j) : \ f(i,j) = k\}$ gives the number of pixel with the gray level equal to k. Based on these probabilities, the digital image f can be considered a discrete random variable

$$p_f = \begin{pmatrix} 1 & 2 & ... & G \\ p_f(1) & p_f(2) & ... & p_f(G) \end{pmatrix} \tag{2}$$

for which $\sum_{k=1}^{G} p_f(k) = 1$. Recall that the cumulative probability distribution of p_f is

$$P_f : \{1,2,...,G\} \rightarrow [0,1],\ P_f(k) = \sum_{l=0}^{k} p_f(l). \tag{3}$$

A more productive approach is to consider the digital image as a continuous random variable $p_f : [0,\infty) \rightarrow [0,1]$ such that $\int_0^\infty p_f(x)dx = 1$. In this case the cumulative probability distribution is

$$P_f : [0,\infty) \rightarrow [0,1),\ P_f(x) = \int_0^x p_f(t)dt.$$

Based on this continuous model the histogram shaping transformation can be defined more easily. Consider that the input digital image f is transformed such that the histogram of the output image g has a shape given by the cumulative distribution $Q : [0,\infty) \rightarrow [0,1)$. The equation that gives histogram shaping is [3]

$$g = Q^{-1}(P_f(f)). \tag{4}$$

The main inconvenience arising form Equation(4) is represented by the inverse function Q^{-1}. Firstly, because the calculation of Q^{-1} might not be easier even for simple shapes. Secondly, we cannot define Q^{-1} for the discrete case therefore it would be difficult to apply (4) to a discrete computation. Perhaps, this is the real reason for seeing no discrete models for histogram shaping. In the following we will propose a discrete model for this transformation.

2 The Superior Smarandache f-Part

In order to propose an equation for the discrete case, we have to find a substitute for Q^{-1}. This is given by the Superior Smarandache f-Part , which represents a direct generalization of the classical ceiling function. Smarandache proposed [5] a generalization of the ceiling function as following. Consider that $f : Z \rightarrow R$ an increasing function such that $lim_{n\rightarrow-\infty} f(n) = -\infty$ and $lim_{n\rightarrow\infty} f(n) = \infty$. The Superior Smarandache f-Part associated with f is $f^{\square} : R \rightarrow Z$ defined by

$$f^{\square}(x) = k \Leftrightarrow f(k-1) < x \leq f(k).$$

Smarandache studied this function in relation to some functions of Number Theory and proposed several conjectures on them [6]. Tabirca also studied the Superior Smarandache f-Part [7] when $f(n) = \sum_{i=0}^{n} i^a$ and proposed equations for $f^{\square}$ when $a = 0,1,2$. Tabirca also applied this function to static parallel loop scheduling [8].

Now, we propose a version of the Superior Smarandache f-Part for our discrete case. Consider that $f : \{1,2,...,G\} \rightarrow (0,1]$ is an increasing function such that $f(G) = 1$. We also consider that this function is extended to 0 with $f(0) = 0$. The Superior Smarandache f-Part associated with f is $f^{\square} : (0,1] \rightarrow \{1,...,G\}$ defined by

$$f^{\square}(x) = k \Leftrightarrow f(k-1) < x \leq f(k),\ \forall x \in [0,1]. \tag{5}$$

This function is also extended in 0 by $f^{\square}(0) = 0$.

Some properties of the function $f^{\square}$ are proposed in the following.

Theorem 1

$$f^{\square}(f(k)) = k,\ \forall k \in \{1,2,...,G\}. \tag{6}$$

Proof The proof is based on the definition of $f^{[]}$ and on the double inequality

$$f(k-1) < f(k) \leq f(k).$$

♠

Theorem 2

$$x \leq f\left(f^{[]}(x)\right) < x + \sup_k (f(k+1) - f(k)), \ \forall x \in (0,1]. \tag{7}$$

Proof

Let us denote $k = f^{[]}(x)$. The definition of $f^{[]}$ provides $f(k-1) < x \leq f(k)$. From this equation, it directly follows that $x \leq f(f^{[]}(x))$.
The second part of Equation (7) comes from the following implication:

$$f(k) < f(k) + x - f(k-1) \Rightarrow$$

$$f\left(f^{[]}(x)\right) < x + \sup_k (f(k) - f(k-1)).$$

♠

Based on these properties, the histogram shaping model of the discrete case is proposed.

3 Histogram Shaping for the Discrete Case

Consider that the input image $f = (f_{i,j} : \ i = 1,2,...,n; \ j = 1,2,...,m)$ is transformed into the output image $g = (g_{i,j} : \ i = 1,2,...,n; \ j = 1,2,...,m)$ such that the histogram of g has a certain shape. Let us presume that the shape of the output histogram is given by the discrete random variable

$$p_h = \begin{pmatrix} 1 & 2 & ... & G \\ p_h(1) & p_h(2) & ... & p_h(G) \end{pmatrix} \tag{8}$$

where $\sum_{k=1}^{G} p_h(k) = 1$.

The general equation of histogram shaping is similar with Equation (4) but $P_h^{[]}$ is used in place of P_h^{-1}. Let consider that the equation of image g is

$$g(i,j) = P_h^{[]}\left(P_f(f(i,j))\right), \ \forall \ (i,j) \in \{1,...,n\} \times \{1,2,...,m\}. \tag{9}$$

We prove that the cumulative probability distribution of g is very close to the cumulative probability distribution of h.

Theorem 3

$$P_g(k) = P_f\left(P_f^{[]}\left(P_h(k)\right)\right), \ \forall k \in \{1,2,...,G\}. \tag{10}$$

Proof The proof is given by the following transformations:

$$P_g(k) = \Pr[g(i,j) \leq k] \ = \ \sum_{l=1}^{k} \Pr\left[P_h^{[]}\left(P_f(f(i,j))\right) = l\right] =$$

$$= \sum_{l=1}^{k} \Pr\left[P_h(l-1) < P_f(f(i,j)) \leq P_h(l)\right] =$$

$$= \sum_{l=1}^{k} \Pr\left[P_f^{[]}(P_h(l-1)) < P_f^{[]}(P_f(f(i,j))) \leq P_f^{[]}(P_h(l))\right] =$$

$$= \sum_{l=1}^{k} \Pr\left[\{f(i,j) \leq P_f^{[]}(P_h(l))\} - \{f(i,j) \leq P_f^{[]}(P_h(l-1))\}\right] =$$

$$= \sum_{l=1}^{k} \left(\Pr\left[f(i,j) \leq P_f^{[]}(P_h(l))\right] - \Pr\left[f(i,j) \leq P_f^{[]}(P_h(l-1))\right]\right) =$$

$$= \sum_{l=1}^{k} \left(P_f\left(P_f^{[]}(P_h(l))\right) - P_f\left(P_f^{[]}(P_h(l-1))\right)\right) =$$

$$= P_f\left(P_f^{[]}(P_h(k))\right) - P_f\left(P_f^{[]}(P_h(0))\right)$$

Since $P_f\left(P_f^{[]}(P_h(0))\right) = 0$ we find that $P_g(k) = P_f\left(P_f^{[]}(P_h(k))\right)$ holds.
♠

From Theorems 2 and 3 the following equation is directly obtained.

$$P_h(k) \leq P_g(k) < P_h(k) + \sup_j (P_f(j+1) - P_f(j)), \ \forall k \in \{0,1,...,G-1\}. \tag{11}$$

Equation (11) provides an estimation of the gap between the quantities $P_h(k)$ and $P_g(k)$. When $\sup_j (P_f(j+1) - P_f(j))$ is smaller these two quantities are very close. Although Equation (11) does not give a perfect equality we can say that the histogram of the image g has the shape very close to h.

The algorithm based on this model firstly finds the functions P_f, P_h and $P_h^{[]}$. Secondly, Equation (9) is applied to obtain the value $g(i,j)$ for each pair (i,j). A full description of this algorithm is presented below.

```
Inputs:
       n,m - the image sizes.
       f=(f[i,j]: i=1,...,n; j=1,...,m) - the input image.
       p_h=(p_h[i]: i=1,...,G) - the desired shape.
Output:
       g=(g[i,j]: i=1,...,n; j=1,...,m) - the input image.
double P_h(int k){
       double s=0;
       if(k<=0 || k>G) return 0;
       for(int i=1;i<=k;i++)s=s+p_h[i];
       return s;
}
int P_h_Inv(double x){
       int k;
       if(x<=0 || x>1) return 0;
```

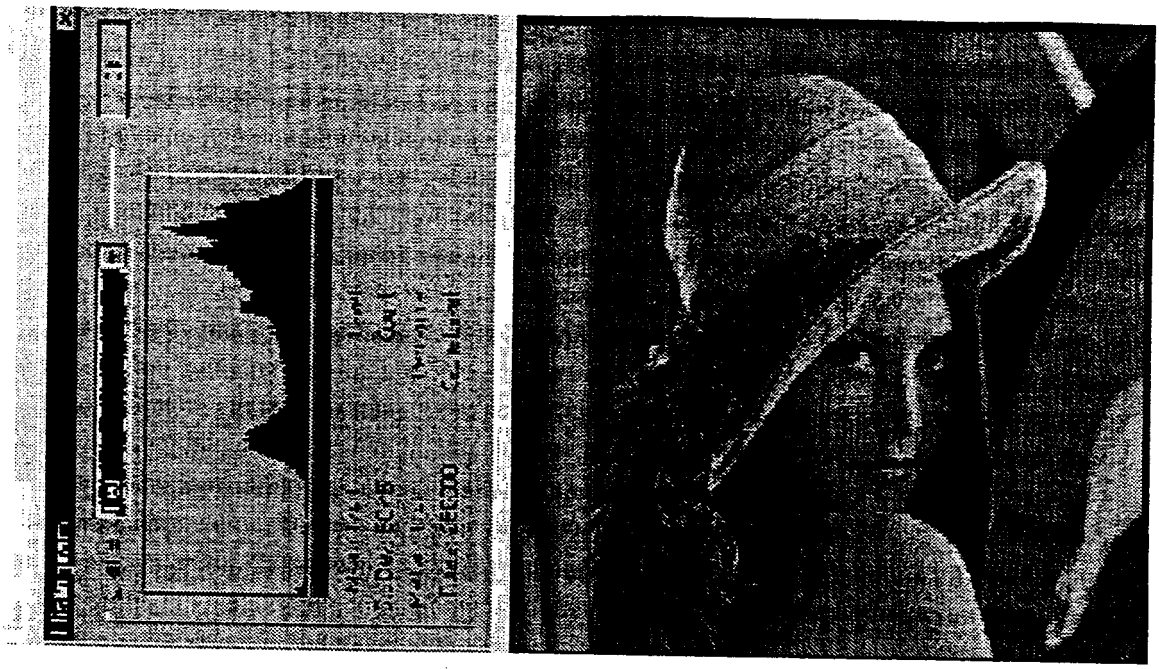

Figure 1: Lenna's picture 1.

```
        for(k=0;P_h(k)<=x;k++);
        return k-1;
}
double P_f(int k){
        double s=0, p_f[G];
        if(k<=0 || k>G) return 0;
        for(int i=1;i<=G;i++)p_f[i]=0;
        for(int i=1;i<=n;i++)for(int j=1;j<=m;j++)p_f[f[i,j]]=p_f[f[i,j]]+1./(n*m);
        for(int i=1;i<=k;i++)s=s+p_h[i];
        return s;
}
int [][] Hist_Shaping(int n, int m, int f[][]){
        int g[n][m];
        for(int i=1;i<=n;i++)for(int j=1;j<=m;j++)
           g[i,j] = P_h_Inv(P_f(f[i,j]));
        return g;
}
```

In order to show that the algorithm performs well we consider an example presented in [1]. Histogram shaping can be used to compare two images of the same scene, which have been taken under different lighting conditions. When the histogram of the first image is shaped to match in the histogram of the second image, the lighting effects might be eliminated.

Consider that we have the images presented Figures 1 and 2. They are two different Lenna's images where the second one has a poor lighting. Each image also contains the histogram for the red channel. The histogram shaping algorithm was applied to transform the second image

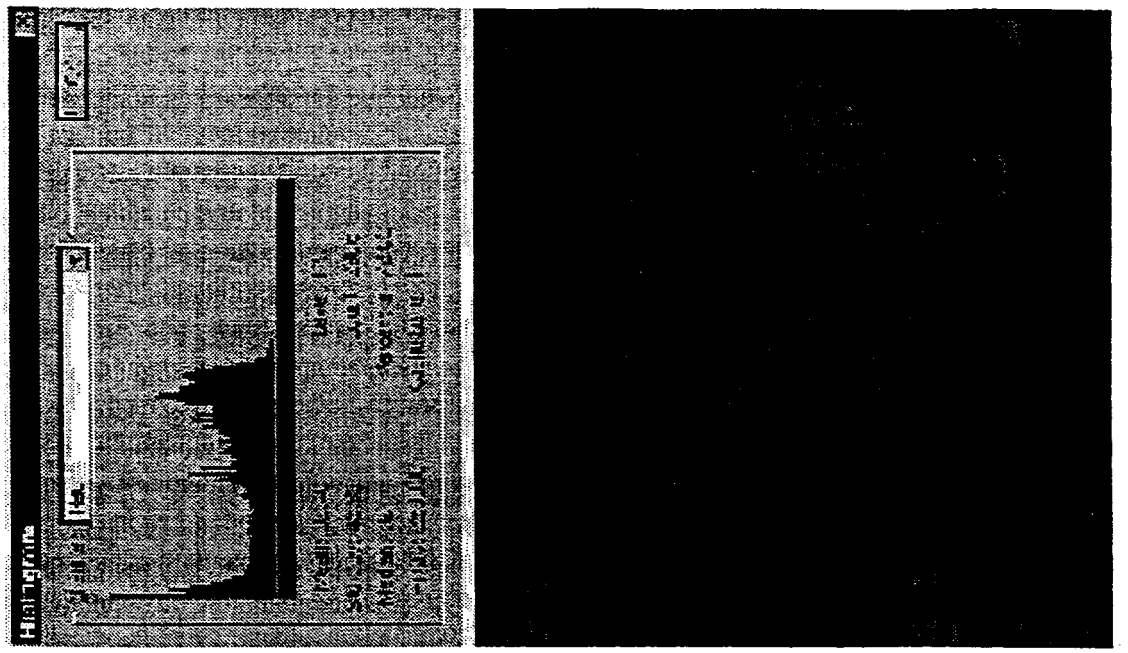

Figure 2: Lenna's picture 2.

according to the histogram of the first image. Figure 3 shows the resulting image which is the same as the first image. Moreover, the histograms of the first and third images are very alike with similar positions for peaks and valleys.

4 Conclusions

This article has introduced a discrete model for the histogram shaping transformation. The model that has been proposed uses the Smarandache ceiling function and is based on the equation $g = P_h^{\square}(P_f(f))$. A example has been also presented in order to prove that the method is viable.

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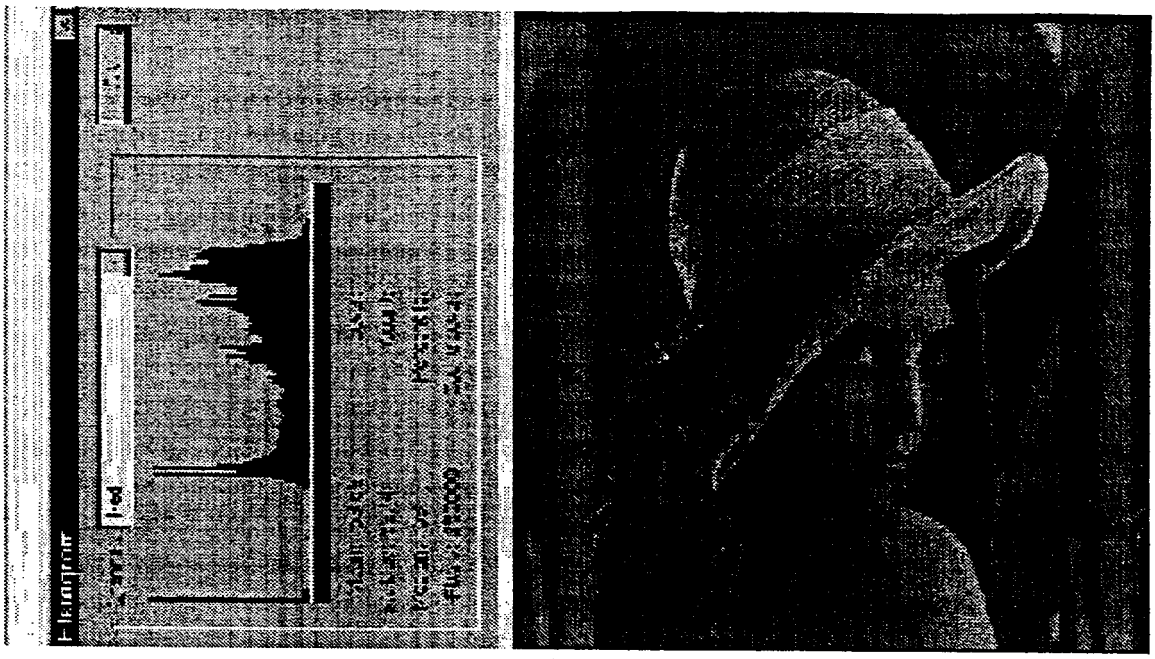

Figure 3: Lenna's Picture after Histogram Shaping.

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On an additive analogue of the function S

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The function S, and its dual S_* are defined by

$$S(n) = \min\{m \in \mathbb{N} : \ n|m!\};$$

$$S_*(n) = \max\{m \in \mathbb{N} : \ m!|n\} \quad \text{(see e.g. [1])}$$

We now define the following "additive analogue", which is defined on a subset of real numbers.

Let

$$S(x) = \min\{m \in \mathbb{N} : \ x \leq m!\}, \quad x \in (1, \infty) \tag{1}$$

as well as, its dual

$$S_*(x) = \max\{m \in \mathbb{N} : \ m! \leq x\}, \quad x \in [1, \infty). \tag{2}$$

Clearly, $S(x) = m$ if $x \in ((m-1)!, m!]$ for $m \geq 2$ (for $m = 1$ it is not defined, as $0! = 1! = 1!$), therefore this function is defined for $x > 1$.

In the same manner, $S_*(x) = m$ if $x \in [m!, (m+1)!)$ for $m \geq 1$, i.e. $S_* : [1, \infty) \to \mathbb{N}$ (while $S : (1, \infty) \to \mathbb{N}$).

It is immediate that

$$S(x) = \begin{cases} S_*(x) + 1, & \text{if} \quad x \in (k!, (k+1)!) \quad (k \geq 1) \\ S_*(x), & \text{if} \quad x = (k+1)! \qquad (k \geq 1) \end{cases} \tag{3}$$

Therefore, $S_*(x)+1 \geq S(x) \geq S_*(x)$, and it will be sufficient to study the function $S_*(x)$.

The following simple properties of S_* are immediate:

1° S_* is surjective and an increasing function

2° S_* is continuous for all $x \in [1,\infty)\setminus A$, where $A=\{k!,\ k\geq 2\}$, and since $\lim\limits_{x\nearrow k!} S_*(x) = k-1$, $\lim\limits_{x\searrow k!} S_*(x) = k$ $(k\geq 2)$, S_* is continuous from the right in $x=k!$ $(k\geq 2)$, but it is not continuous from the left.

3° S_* is differentiable on $(1,\infty)\setminus A$, and since $\lim\limits_{x\searrow k!} \dfrac{S_*(x)-S_*(k!)}{x-k!} = 0$, it has a right-derivative in $A\cup\{1\}$.

4° S_* is Riemann integrable in $[a,b]\subset\mathbb{R}$ for all $a<b$.

a) If $[a,b]\subset[k!,(k+1)!)$ $(k\geq 1)$, then clearly

$$\int_a^b S_*(x)dx = k(b-a) \tag{4}$$

b) On the other hand, since

$$\int_{k!}^{l!} = \int_{k!}^{(k+1)!} + \int_{(k+1)!}^{(k+2)!} + \ldots + \int_{(k+l-k-1)!}^{(k+l-k)!}$$

(where $l>k$ are positive integers), and by

$$\int_{k!}^{(k+1)!} S_*(x)dx = k[(k+1)!-k!] = k^2\cdot k!, \tag{5}$$

we get

$$\int_{k!}^{l!} S_*(x)dx = k^2\cdot k! + (k+1)^2(k+1)! + \ldots + [k+(l-k-1)]^2[k+(l-k-1)!] \tag{6}$$

c) Now, if $a\in[k!,(k+1)!]$, $b\in[l!,(l+1)!)$, by

$$\int_a^b = \int_a^{(k+1)!} + \int_{(k+1)!}^{l!} + \int_{l!}^{k}$$

and (4), (5), (6), we get:

$$\int_a^b S_*(x)dx = k[(k+1)! - a] + (k+1)^2(k+1)! + \ldots +$$

$$+[k+1+(l-k-2)]^2[k+1+(l-k-2)!] + l(b-l!) \tag{7}$$

We now prove the following

Theorem 1.

$$S_*(x) \sim \frac{\log x}{\log\log x} \quad (x \to \infty) \tag{8}$$

Proof. We need the following

Lemma. *Let* $x_n > 0$, $y_n > 0$, $\frac{x_n}{y_n} \to a > 0$ *(finite) as* $n \to \infty$, *where* $x_n, y_n \to \infty$ $(n \to \infty)$. *Then*

$$\frac{\log x_n}{\log y_n} \to 1 \quad (n \to \infty). \tag{9}$$

Proof. $\log \frac{x_n}{y_n} \to \log a$, i.e. $\log x_n - \log y = \log a + \varepsilon(n)$, with $\varepsilon(n) \to 0$ $(n \to \infty)$. So

$$\frac{\log x_n}{\log y_n} - 1 = \frac{\log a}{\log y_n} + \frac{\varepsilon(n)}{\log y_n} \to 0 + 0 \cdot 0 = 0.$$

Lemma 2. a) $\frac{n \log\log n!}{\log n!} \to 1$;

b) $\frac{\log n!}{\log(n+1)!} \to 1$;

c) $\frac{\log\log n!}{\log\log(n+1)!} \to 1$ as $n \to \infty$ (10)

Proof. a) Since $n! \sim Ce^{-n}n^{n+1/2}$ (Stirling's formula), clearly $\log n! \sim n\log n$, so b) follows by $\frac{\log n}{\log(n+1)} \sim 1$ ((9), since $\frac{n}{n+1} \sim 1$). Now c) is a consequence of b) by the Lemma. Again by the Lemma, and $\log n! \sim n\log n$ we get

$$\log\log n! \sim \log(n\log n) = \log n + \log\log n \sim \log n$$

and a) follows.

Now, from the proof of (8), remark that

$$\frac{n\log\log n!}{\log(n+1)!} < \frac{S_*(x)\log\log x}{\log x} < \frac{n\log\log(n+1)!}{\log n!}$$

and the result follows by (10).

Theorem 2. *The series* $\sum_{n=1}^{\infty} \frac{1}{n(S_*(n))^\alpha}$ *is convergent for* $\alpha > 1$ *and divergent for* $a \leq 1$.

Proof. By Theorem 1,

$$A\frac{\log n}{\log\log n} < S_*(n) < B\frac{\log n}{\log\log n}$$

$(A, B > 0)$ for $n \geq n_0 > 1$, therefore it will be sufficient to study the convergence of $\sum_{n\geq n_0}^{\infty} \frac{(\log\log n)^\alpha}{n(\log n)^\alpha}$.

The function $f(x) = (\log\log x)^\alpha / x(\log x)^\alpha$ has a derivative given by

$$x^2(\log x)^{2\alpha} f'(x) = (\log\log x)^{\alpha-1}(\log x)^{\alpha-1}[1 - (\log\log x)(\log x + \alpha)]$$

implying that $f'(x) < 0$ for all sufficiently large x and all $\alpha \in \mathbb{R}$. Thus f is strictly decreasing for $x \geq x_0$. By the Cauchy condensation criterion ([2]) we know that $\sum a_n \leftrightarrow \sum 2^n a_{2^n}$ (where $\leftrightarrow$ means that the two series have the same type of convergence) for (a_n) strictly decreasing, $a_n > 0$. Now, with $a_n = (\log\log n)^\alpha / n(\log n)^\alpha$ we have to study $\sum \frac{2^n(\log\log 2^n)^\alpha}{2^n(\log 2^n)^\alpha} \leftrightarrow \sum \left(\frac{\log n + a}{n+b}\right)^\alpha$, where a, b are constants ($a = \log\log 2$, $b = \log 2$). Arguing as above, (b_n) defined by $b_n = \left(\frac{\log n + a}{n+b}\right)^\alpha$ is a strictly positive, strictly decreasing sequence, so again by Cauchy's criterion

$$\sum_{n\geq m_0} b_n \leftrightarrow \sum_{n\geq m_0} \frac{2^n(\log 2^n + a)^\alpha}{(2^n+b)^\alpha} = \sum_{n\geq m_0} \frac{2^n(nb+a)^\alpha}{(2^n+b)^\alpha} = \sum_{n\geq m_0} c_n.$$

Now, $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \frac{1}{2^{\alpha-1}}$, by an easy computation, so D'Alembert's criterion proves the theorem for $\alpha \neq 1$. But for $\alpha = 1$ we get the series $\sum \frac{2^n(nb+a)}{2^n+b}$, which is clearly divergent.

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HIPÓTESE DE SMARANDACHE: EVIDÊNCIAS, IMPLICAÇÕES E APLICAÇÕES[1]

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RESUMO: Em 1993, Smarandache propôs que não há uma velocidade limite na natureza, baseado no paradoxo EPR-Bell (Einstein, Podolsky, Rosen, Bell). Embora pareça que este paradoxo foi resolvido recentemente, ainda há várias outras evidências que nos guiam a acreditar que a hipótese de Smarandache está correta na mecânica quântica e até nas teorias de unificação. Se a hipótese de Smarandache revelar-se verdadeira em qualquer circunstância, alguns aspectos da física moderna terão que se "reajustar" para concordar com a hipótese de Smarandache. Em adição, quando o significado da hipótese de Smarandache tornar-se totalmente entendido, uma revolução na tecnologia, especialmente nas comunicações, irá surgir.

I. EVIDÊNCIAS DE FENÔMENOS SUPERLUMINAIS E A HIPÓTESE DE SMARANDACHE

Aparentemente foi Sommerfeld quem primeiro notou a possível existência de partículas mais rápidas que a luz, mais tarde chamadas de *tachyons* por Feinberg [1]. Todavia, tachyons possuem massa imaginária, assim nunca foram detectados experimentalmente. Por massa imaginária entendemos uma massa proibida pela teoria da relatividade. Entretanto, a relatividade não proíbe diretamente a existência de partículas superluminais *sem massa*, como o fóton, mas sugere que fenômenos superluminais culminariam em viagem no tempo. Então, muitos físicos assumiram que fenômenos superluminais não existem no universo, outra forma teríamos que explicar todos os paradoxos do tipo "mate o seu avô" [2]. Um famoso exemplo de paradoxo deste tipo é o problema de causalidade.

[1] A ser publicado, *Smarandache Hypothesis: Evidences, Implications and Applications*, em "Smarandache Notions Journal", Vol. 12, 4, University of New Mexico, à convite de Dr. Minh Perez.

Mesmo assim, a mecânica quântica sugere que movimentos superluminais existem. De fato, há hipóteses da existência obrigatória de movimentos superluminais na natureza [3, 4]. O paradoxo EPR-Bell é o mais famoso exemplo. Ponderando sobre este paradoxo, Smarandache propôs em 1993, em uma palestra ao Brasil, que não há tal velocidade limite no universo, como postulado por Einstein [5]. Parece que este paradoxo foi recentemente resolvido por L. E. Szabó [6]. Mesmo assim, ainda há várias evidências de velocidades infinitas (comunicação instantânea) no universo, como veremos brevemente.

I.1. A Teoria de Rodrigues-Maiorino

Estudando soluções das equações de Maxwell e Dirac-Weyl, Waldyr Rodrigues Jr. e José Maiorino foram capazes de propor uma teoria unificada para construção de velocidades arbitrárias na natureza (por arbitrária entende-se $0 \leq v \leq c$) em 1997 [1]. Eles também sugeriram que não há tal velocidade limite no universo, assim a hipótese de Smarandache pôde ser promovida a teoria como teoria de Smarandache-Rodrigues-Maiorino (SRM).

Fato único da teoria de Rodrigues-Maiorino é que o princípio da relatividade especial sofre uma quebra, entretanto, mesmo construções relativísticas da mecânica quântica, como a equação de Dirac, concordam plenamente com fenômenos superluminais. De acordo com a teoria de Rodrigues-Maiorino, até mesmo um conjunto bem posicionado de espelhos pode acelerar uma onda eletromagnética a velocidades superiores a da luz. Essa afirmação foi mais tarde confirmada por Saari e Reivelt (1997) [8], que produziram uma onda X (nomeada desta forma por J. Y. Lu, um contribuidor de Rodrigues) usando uma lâmpada de xenônio interceptada com um conjunto de lentes e orifícios.

A teoria SRM é uma construção matemática pura e forte da equação de onda relativística que nos indica que não há nenhuma velocidade limite no universo.

I. 2. Experimentos Superluminais

Muitos experimentos, principalmente modos evanescentes, resultam em propagação superluminal. O primeiro modo evanescente bem sucedido foi obtido em 1992 por Nimtz [9]. Nimtz produziu um sinal $4.34c$, e mais tarde um sinal FM $4.7c$ com a 40^a sinfonia de Mozart. Esse sucesso de Nimtz seria mais tarde superado por outros resultados até 8 vezes mais rápidos que a constante c.

No caso do experimento de Nimtz não está claro se ele viola o paradoxo de causalidade. Em contrapartida, L. J. Wang, A. Kuzmich e A. Dogariu recentemente publicaram um extraordinário resultado de dispersão anômala o qual um pulso de luz foi acelerado 310 ± 5 vezes a velocidade da luz, sem violar o paradoxo de causalidade, portanto resultando em viagem no tempo! Na prática, isto significa que um pulso de luz propagando pela célula de vapor atômico aparece na saída muito antes de ter propagado a mesma distância no vácuo e o pico do pulso parece sair da célula antes mesmo de entrar [10].

I.3. Revisão da Velocidade da Gravidade

A teoria geral da relatividade postula que a velocidade da gravidade é a mesma que a constante *c*. Porém, se a velocidade da luz não é a velocidade limite do universo, não seria hora de revisar este postulado? Van Flandern publicou alguns resultados astrofísicos que indicam que a gravidade é superluminal [11]. Observações de algumas galáxias feitas pela NASA sugerem que algumas galáxias estão girando com velocidade superluminal [12].

Os dados de Van Flandern foram mais tardes explicados por uma teoria que não usava de movimentos superluminais por Ibison, Puthoff e S. R. Little [13]. Ainda assim, observações de sinais superluminais vindo de galáxias permanece inexplicadas pelo ponto de vista subluminal.

I.4. Tachyons

Alguns modelos da teoria de supercordas, nossa mais promissora candidata para teoria unificada da física, inclui tachyons, as partículas hábeis de viajar mais rápido que a luz. Mesmo assim, físicos encontraram uma maneira de "cortar" a teoria de maneira que as soluções de tachyons desaparecem; alguns outros, como Freedman, defendem que a teoria de supercordas não deveria ser cortada de tal forma em absoluto [1]. A teoria das supercordas é provavelmente o melhor campo para o estudo de tachyons, pois não irá forçar o uso do artifício de massa imaginária. Prof. Michio Kaku comparou a idéia de mais dimensões em física a um esquema de matrizes em seu *Hyperspace*. Imagine uma matriz 4x4 a qual temos dentro a teoria da relatividade e outra matriz 4x4 onde temos o Modelo Padrão. Se construirmos uma matriz maior, digamos 8x8, seriamos capaz então de incluir ambas a mecânica quântica e a relatividade em uma única matriz. Esta é a idéia principal de unificação a partir da adição de mais dimensões. Da mesma maneira, trabalhando apenas com matrizes 4x4 não temos espaço suficiente para trabalhar com tachyons. Todavia, em uma matriz maior teríamos o espaço necessário para encontrar construções sólidas de modelos de tachyons.

Tachyons já foram, de uma maneira obscura, observados em chuveiros de ar de raios cósmicos.

II. Implicações e Aplicações

De acordo com a teoria de Rodrigues-Maiorino a conseqüência da existência de fenômenos superluminais seria a quebra do princípio da relatividade, mas não precisaríamos alterar nada na mecânica quântica. Mais precisamente, nos parece que é a mecânica quântica quem está banindo a antiga teoria relativística segundo a teoria SRM. Apesar disso, na realidade a teoria da relatividade aceita algum tipo de comunicação superluminal que resulta em viagem no tempo, como Wang e seus contribuidores mostraram.

Talvez nós poderemos, num futuro distante, enviar mensagens para o futuro ou passado. De qualquer forma, fenômenos superluminais teriam uma aplicação mais realista com comunicação local, pois de acordo com a teoria Rodrigues-Maiorino, a onda X é *fechada* no sentido que ela não perde energia enquanto viaja. Então, uma mensagem de rádio superluminal de onda X chegaria a seu destino com quase a mesma condição em que foi enviada e ninguém, exceto o destino, poderia espiar o conteúdo da mensagem. A invenção de um tal transmissor superluminal seria de grande poder associado a pastilha para desviar a luz em 90° do MIT na manufatura de fibras ópticas.

III. Conclusão

Os vários experimentos e teorias sólidas que nascem da mecânica quântica envolvendo fenômenos superluminais são alto-níveis de indicação da hipótese de Smarandache que não há tal velocidade limite na natureza. Isto implica em uma quebra do postulado da relatividade de Einstein, mas não em nenhum campo da mecânica quântica, até mesmo na função de onda relativística. Como em nossa evolução chegou um tempo em que a mecânica newtoniana não era suficiente para compreender alguns novos aspectos da natureza, talvez se aproxima um tempo em que a teoria da relatividade de Einstein deve ser deixada de lado, pois então a mecânica quântica irá governar.

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SUPERLUMINALS AND THE SPEED OF LIGHT

Jason WRIGHT*

Abstract: This brief paper was submitted as partial requirement for a Chemistry course. The topic was recommended to Dr. Kamala Sharma*.

Key Concepts: superluminals, locality/nonlocality, mechanistic/ nonmechanistic, Smarandache Hypothesis.

Definitions:

Superluminals are phenomena capable of greater than light speed.

Locality is the assumption that change in physical systems requires presence of mechanistic links between cause and effect.

Nonlocality is that which is displayed by physical systems in which change evidently happens without such mechanical links.

Mechanistic is direct physical contact (push-and-pull interactions) between parts of dynamic systems characteristic of machines.

Nonmechanistic is nonphysical interaction between parts of a dynamic system characteristic of superluminals.

For more than a century, an argument has been carried on concerning which is a more accurate picture, or model, of the workings of the universe. Basic to this argument is the difference between the view of the world presented to us by classical (Newtonian) physics and quantum physics. Classical physics held sway on a macroscopic scale until Max Planck discovered that on the very small scale, quantum mechanics was more accurate than classical mechanics could provide. Central to this argument were

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physicists like Planck, Einstein, Hizenburg, Scheodinger, Bohr, Bohm, and a number of others. Most basic to this controversy is how action at a distance can occur, e.g., how does the sun hold the planets in place without any mechanical means of doing so. Einstein did not agree with Newton's theory of gravitation, because there was no evidence, and still isn't, of any force acting across space to hold the planets near the sun. Einstein developed his own theory of gravitation to give a much more mechanical view of gravitation. The sun influences the space, warps the space, near it, so that the planets roll around the sun much as marbles would roll around a tightly stretched sheet with some sort of indentation in the middle of it.

Einstein's theory of gravitation was as good or better than Newton's, however, on the subatomic level, motions could not be accounted for accurately without a new theory: Quantum Mechanics. With Quantum physics a new wrinkle was added to the discussion. It appeared that particles could communicate at a greater than light speed. Einstein thought this possibility absurd, and he and a couple of his assistants came up with a thought experiment (EPR) to refute the possibility that speeds greater than light could occur. Being convinced that the speed of light was the top speed of the universe, Einstein imagined two particles with opposite spins could change their relative spins only if somehow they communicated at greater than the speed of light. Since he had already absolutely accepted the speed of light as the maximum velocity in the universe, he had to conclude that this instantaneous communication between the spinning particles was absurd, or absolutely impossible. Seems like sort of a circular argument.

Paralleling this mechanistic/nonmechanistic debate was the concept of locality/ nonlocality. Local was used as synonymous with mechanical and nonlocality with non-mechanical. Bell argued that if we could show that the notion of "local" did not exist at the subatomic level, then speeds seemingly occurring at greater than light would be explainable. I.e., if some things in the universe are really nonlocal, then communication could occur instantly, because they would not involve time or space. These instantaneous messengers came to be called superluminals. Bell's experiments proved the existence of superluminals, and, hence, Bohr's view of mechanics was proven right, and Einstein's view wrong. There can be nonmechanistic action at a distance at the subatomic level, if you can show some sort of communication without regard for time and space.

In our macroscopic world we live in a universe of "locality" but on the subatomic, microscopic world, all localities can be taken as the same locality, and, therefore, non-local. On a large scale our world seems to be very mechanistic, i.e., things have to touch and move things through space and time for anything to happen, whereas, on the small scale, subatomic level, things are still capable of behaving as they did at the big bang,

i.e., they all were at the same place at the same time: All places were one place and all times
were one time. Therefore, if subatomic particles have retained their big bang behavior, and experiments are showing that they do, then these particles are communicating at faster than light velocities, because they don't have to traverse any time or space. Super-luminal communication does seem to be possible, i.e., communication unrelated to any particular velocity.

Dr. Florentin Smarandache argues in his paper, "There Is No Speed Barrier In The Universe," called "Smarandache Hypothesis," that paired entangled particles (photons) communicate instantly concerning their individual states, i.e., measuring one immediately determines the measurement of the other no matter how far separated. His conclusion had to be that this sort of subatomic particle behavior must be taken as sound evidence that, on the quantum level, there is no restraining finite speed. Even after Bell's inequality experiment, which extended the Einstein – Podolsky – Rosen (EPR) thought experiment, that has shown conclusively that there has to be phenomena interacting at greater than light speed, there is criticism of Dr. Smarandache's paradox. The criticisms go like this: "While it is true that modern experiments have demonstrated the existence of types of measurable superluminal phenomena, none of these experiments are in conflict with causality or special relativity since no information or physical object actually travels at speeds greater than light to produce the observed phenomena." It seems easy enough for these criticisms to say "no information" is moving from particle to particle or that these particles are not "physical objects," but, then, what is happening between them, and what are they. The point is that something is occurring at greater than light speed, called "superluminals," and it has been measured. It may be better for us to say that there is some sort of "interaction" between subatomic particles happening at greater than light speed, however, whatever we call it, it exists, and, therefore, we have to amend our view of light speed as the maximum universal velocity.

Various experimental apparatus were designed essentially with the same premise, that of splitting up certain qualities or characteristics of different kinds of particles. The results at the detector were startling and very difficult to explain unless at the quantum level one assumed communication, or some sort of interaction, between the particles at a greater than light speed. John Stewart Bell's experiments have swept away the assumption, on the microscopic level, because we now have proof. However, that has been an enormous, almost overwhelming discovery, because it shows us that nature can behave in a totally noncommonsensical manner. Things do work on one another without touching and without regard for time or space. This finding has been abhorrent to many physicists, including Einstein, however, he was wrong in his belief in a totally mechanistic world. A great deal of our world is quite concealed from us, and our lab work on it, and our mathematics, reveal that in the very small subatomic world, things behave according to laws and a logic very different from the laws and logic of the very large world of people, and, planets, and galaxies. This is difficult for many of us to accept, but assumptions were made about the operations of the universe, and some of these assumptions are being shown wrong. In a similar way we believed for a very long time, we assumed, that the earth was the center of our solar system. We now have to alter our thinking relative to another fundamental matter.

An even more crucial area of concern relative to the issue of superluminals, a much more fundamental area of physics that was illuminated by experiments developed for testing for the possibly of superluminals, is the ongoing debate over whether the universe is a totally mechanistic one (classical/Newtonian/local) or is it in some sense non-mechanistic (nonlocal). John Stewart Bell, an Irish physicist, worked out

experiments to test the classical assumption that nature works in a strictly "local," mechanistic way. The results of these experiments revealed that the classical assumption was wrong – nature is in some sense nonlocal (nonmechanistic), and, hence, the possibility of effects occurring between subatomic particles at a speed greater than light, is quite real. David Bohm, another physicist who spent much of his life studying this surprising side of nature, remarked near the end of his life, "Quantum strangeness is a keyhole through which we have caught a first glimpse of another side of nature, one in which the universe is neither deployed across vast reaches of space and time nor harbors many "things". Rather it is one, interwoven thing, which incorporates space and time but in some sense subordinates them (e.g. superluminals) perhaps by treating them as important but non-fundamental aspects of the interface between the universe and the observer who investigates it."

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Faster than light?

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Abstract

The hypothesis formulated by Smarandache on the possibility that no barriers exist in the Universe for an object to travel at any speed is here shortly analyzed.

In 1980, contrarily to what previously postulated by Einstein, Smarandache, according to the EPR paradox, formulated the hypothesis that no barriers exist in the Universe for an object (or particle or information or energy) to travel at any speed.

Recently, EPR-type experiments (entanglement and tunneling) have been carried out which prove that quantum mechanics is "non-local" and that the speed of light can be overcome. In fact, these experiments have highlighted that "space-time" separated systems, which previously had mutually interacted, are anyhow connected and such a connection is instantaneous, discriminating and not attenuated.

Instantaneousness is not new in Physics. It suffices mentioning the Newtonian Physics where the instantaneousness of the force of gravity is hypothesized. However, gravity decreases with the square of distance and such an interaction acts on all of the masses in the Universe, contrarily to what "non-local" mechanics seems to do.

If it is possible to travel at speeds greater than that of light, is it possible to exchange information faster than the speed of light? The answer is no. It has been demonstrated that even if it was possible to transmit information into the space at speeds greater than that of light the receiver is not capable to correctly reconstruct the sent information. Therefore, the Einstein principle of causality is not violated; consequently, it is not possible to detect an effect earlier than its cause.

To exchange information at speeds greater than that of light, the Schroedinger's equation must be admitted slightly non-linear. So far, all of the dedicated experiments have proved exactly the contrary. However, the Physics of Chaos has highlighted that nature, which until a few years ago

was thought to be linear, prefers instead showing itself through highly non-linear phenomena and that the linear ones constitute a rare exception. So, why shall we believe that, on an atomic scale where quantum mechanics applies, nature should follow linear relationships ?

Moreover, is it actually true that relativity prevents from any possibility that objects exist which travel at a speed greater than that of light ? Actually, relativity states that it is not possible to accelerate an object up to a speed greater than that of light since this would need to rely on all of the energy of the Universe; in fact, as the speed of the object increases its mass gets greater and greater. On the other hand, relativity does not prevent the possibility that objects exist with a speed greater than that of light, such as in specific reactions where tardions ($v<c$) can originate tachions ($v>c$). In such a case, a particle does not need to be accelerated to a speed greater than that of light since it already exists with a speed greater than "c". The only problem with tachions is that these hypothetical particles should posses an imaginary mass that is too strong of an assumption from a physical point of view. Several unsuccessful experiments have been carried out so far with the aim to find tachions (i.e., through the attempt to detect Cerenkov's radiation that should be emitted by the ones that travel at a speed greater than that of light)

This might mean that:

1) tachions do not exist
2) tachions interact only rather weakly with matter (capture rate less than that of neutrinos) and therefore it is complicated to detect them.
3) Necessary energies to generate tachions are too high for the performances of present accelerators.

In summary, the emerging "non-local" quantum mechanics seems to con-validate the Smarandache Hypothesis, without violating Einstein's "causality" principle. The relativity theory will need to be completely re-written if proofs are brought that Schroedinger's equation is weakly "non-linear"; in which case, information could be transmitted faster than light.

Unfortunately, it is not possible to resolve this dispute for the moment; all of the hypothesis remain still valid.

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